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**ASPECTS OF MECHANICAL BEHAVIOR OF ROCK  
UNDER STATIC AND CYCLIC LOADING. PART A.  
MECHANICAL BEHAVIOR OF ROCK UNDER STATIC  
LOADING**

**Jesus Basas, et al**

**Wisconsin University**

**Prepared for:**

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Advanced Research Projects Agency**

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## ASPECTS OF MECHANICAL BEHAVIOR OF ROCK UNDER STATIC AND CYCLIC LOADING

### Part A Mechanical Behavior of Rock Under Static Loading

Semi-Annual Technical Progress Report  
September 1972

Sponsored by Advanced Research Projects Agency  
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**ASPECTS OF MECHANICAL BEHAVIOR OF ROCK UNDER  
STATIC AND CYCLIC LOADING**

**PART A: MECHANICAL BEHAVIOR OF ROCK UNDER STATIC LOADING**

**Semi-Annual Technical Progress Report  
September 1972**

by

**R. W. Heins (Co-Principal Investigator  
with B. C. Haimson)**

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<u>11. ABSTRACT</u> <p>Brazilian tests were carried out to determine size-tensile strength dependence in dacite, Valders limestone, and St. Cloud gray granodiorite. It was concluded that there is a size effect on tensile strength; in Brazilian test, this effect is governed mainly by the position and orientation of the internal flaws relative to the loaded diametral plane.</p> <p>A two-dimensional computer program simulating the Brazilian test was written. The program employs four-sided, isoparametric elements and is based on the same failure criteria used in the first annual progress report. The present program, however, is more efficient and contains several features not found in the first program. Test runs have proven that the program can predict accurately the progression of failure in Brazilian test and, to a lesser extent, the correlation of load and displacement.</p> <p>A program employing both two-dimensional and three-dimensional elements is proposed. The program will be based on more realistic failure criteria and will take into account rock anisotropy.</p>		

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this document may be better  
studied on microfilm

Security Classification

IA

finite element analysis  
Brazilian test  
indirect tensile strength  
model of rock failure  
force-displacement curve  
bounded distributions of strength and  
stiffness  
finite element progressive failure model

*Ib*

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Security Classification

## PREFACE

This report covers the first six months accomplishments in the research program entitled, "Mechanical Behavior of Rock Under Static Loading," R. W. Heins, Co-Principal Investigator. The program is Part A of the project entitled, "Aspects of Mechanical Behavior of Rock Under Static and Cyclic Loading" (Contract No. H020041). Part B of the project is published in a separate volume.

## SUMMARY

### ASPECTS OF MECHANICAL BEHAVIOR OF ROCK UNDER STATIC LOADING

#### PART A

##### Summary of Work to Date

Brazilian tests were carried out on three rocks (dacite, Valders limestone, and St. Cloud gray granodiorite) to determine size-tensile strength dependence. Plots of tensile strength versus specimen dimension (length or diameter) are shown in Chapter 1. It was concluded that there is a size effect on tensile strength; in Brazilian test, this effect is governed mainly by the position and orientation of the internal flaws relative to the loaded diametral plane rather than by the extent and number of the flaws.

A two-dimensional computer program simulating the Brazilian test has been completed. The program employs four-sided, isoparametric elements and is based on the same failure criteria described in the first annual progress report. The present program, however, is more efficient and contains several features not found in the first program. Test runs have proven that the program can predict accurately the progression of failure in Brazilian test and, to a lesser extent, the correlation of load and displacement.

Development of a program employing both two-dimensional and three-dimensional elements has started. The program will be based on more realistic failure criterion and will take into account rock anisotropy. Most of the writing of the program has been done and debugging of the program is in progress.

### Future Work

Development of the combined two- or three-dimensional program will continue. Several examples will be run to check the correctness of the program.

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**PART A**

**ASPECTS OF MECHANICAL BEHAVIOR OF ROCK  
UNDER STATIC AND CYCLIC LOADING**

**A FINITE ELEMENT MODEL OF ROCK FAILURE  
FROM THE BRAZILIAN TEST**

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A. Hayatdavoudi  
Robert W. Heins**

**September 1972**

**Department of Metallurgical & Mineral Engineering  
The University of Wisconsin  
Madison, Wisconsin 53706**

# CHAPTER 1

## SIZE EFFECT ON BRAZILIAN TEST

### 1.1 Introduction

When applying values of mechanical properties of rocks obtained from laboratory tests to actual problems, it is essential for reasons of safety and economy that size effect, if any, be established. A design, for example, that does not take into account size effect could be unsafe if, in fact, size effect exists. On the other hand, a design based on the existence of size effect could be overly conservative if no such effect exists.

Although considerable experimental work has been undertaken to determine size effect in rocks, the findings have so far been inconclusive and often contradictory. In tests to study size-strength dependence, it has been observed (2) that, with increasing size, strength either (a) decreases, (b) remains unchanged, or (c) increases. A very logical explanation of these widely divergent size effects was offered by Koifman (3). He hypothesized that size effect is governed by two factors, namely: natural internal imperfections which he called "volume" factor and "changes in the surface layers, brought about by mechanical, physical or chemical action, or by influences of the environment" which he called "surface" factor. Koifman claimed that with increased size, strength will decrease when the "volume" factor is dominant, could increase if the "surface" factor is dominant, or will remain the same if the two factors balance each other. He went on to say that under tensile stresses the "volume" factor will always prevail and hence, the strength of the rock will always decrease with increasing size. He based his argument on the assumption that the number and extent of the internal flaws increases with size. Although this assumption has a statistical basis, it might not be valid in actual situations.

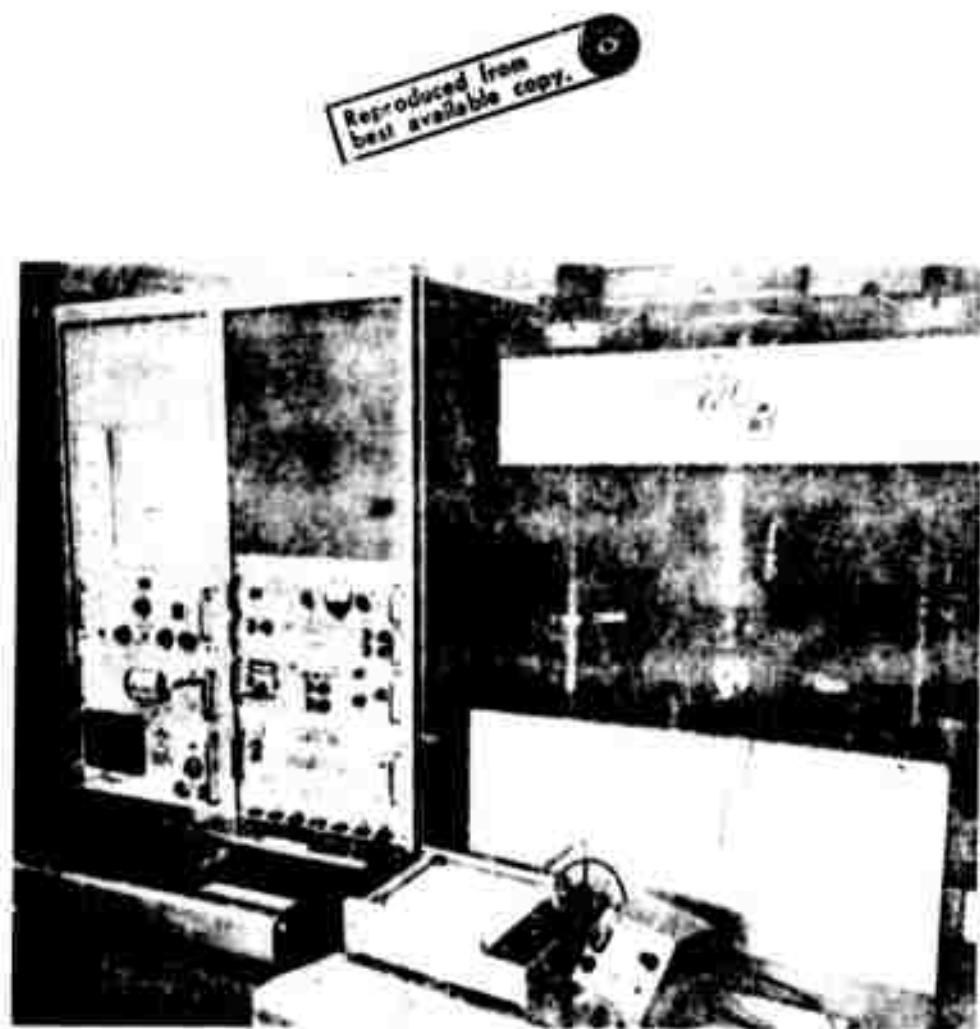
Indeed, in rock masses where nonhomogeneity occurs more often than not a sample taken from a relatively defect-free region could easily contain much fewer structural defects than smaller samples taken from other regions not quite as defect-free. This is particularly true for small-size samples such as those used in laboratory tests. Furthermore, in Brazilian tests the tensile strength could be much more sensitive to the position and orientation of the structural defects relative to the loading axis than the number and extent of these defects. Thus, in Brazilian tests at least, the possibility of strength increasing with size should not be ruled out. Several investigators (2, 3) have reported such a size-strength variation.

In the present study, an attempt will be made to correlate specimen size and tensile strength as obtained from Brazilian test. Three types of rocks were tested. These are Valders limestone, St. Cloud gray granodiorite, and dacite.

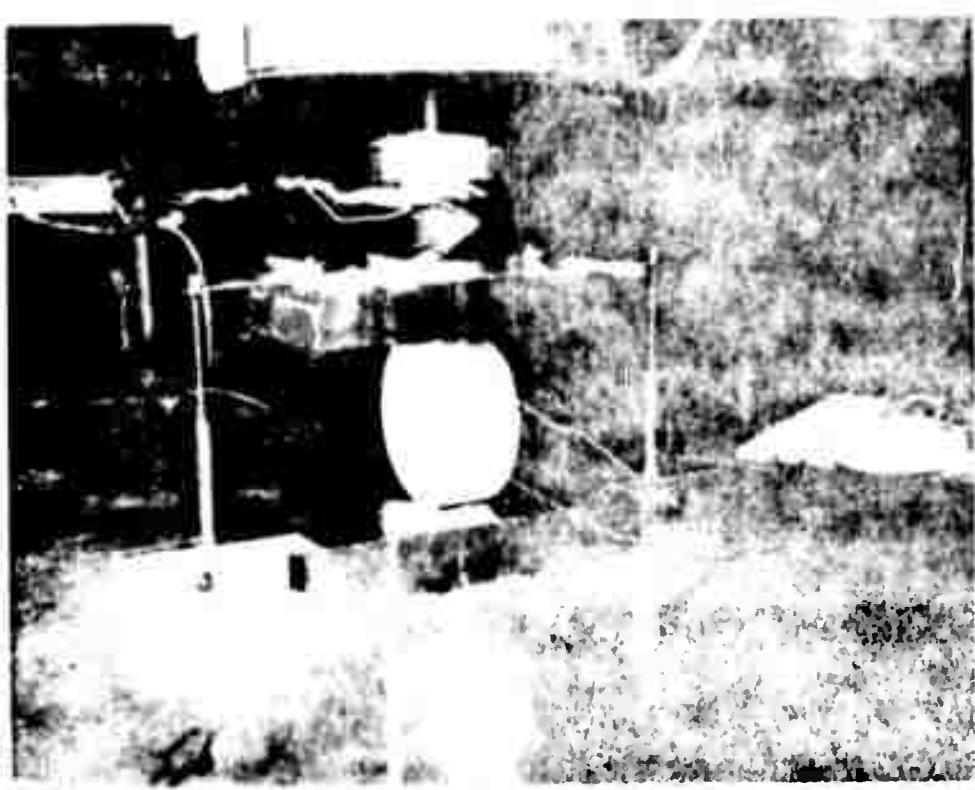
## 1.2 Experimental Procedure and Results

The tests were carried out on a MB Universal Testing Machine (Fig. 1.1) according to the procedure described in the first annual progress report (1). The specimens were 1", 2" and 3" in diameter and 1/2", 1" and 2" in length (or thickness), with all nine possible combinations of length and diameter represented. The specimens were tested in random order and without due regard as to which diametral axis would be loaded. At least three samples of each size were tested. A typical Brazilian test set-up is shown in Fig. 1.2.

Two modes of testing, namely, stress-controlled mode and strain-controlled mode, were used. In the stress-controlled mode, the load is applied at a predetermined constant rate of approximately 100 pounds per second. In the strain-controlled mode, the load is applied at a varying rate depending on the lateral strain at the center of the specimen. In the latter mode the load can be reduced faster than the specimen breaks thus avoiding the catastrophic failure which characterizes stress-controlled tests.



**Figure 1.1** The MB Universal Testing Machine



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Figure 1.2      Brazzle in West

It is thus possible to obtain complete stress-strain curves in strain-controlled tests.

To make the two loading modes equivalent during the early stage of loading, the strain-controlled mode was programmed to provide a maximum strain of 10,000 micro-inch per inch in 800 seconds. This rate, it was estimated, is approximately equal to the 100 lbs./sec. rate used in the stress-controlled mode.

Plots of tensile strength ( $\sigma_t$ ) versus specimen dimensions for all rocks are shown in Figs. 1.3 through 1.14. The curves were plotted on the basis of the equation,

$$\sigma_t = \theta_1 L^2 D^{-3}$$

in which L is length; D is diameter; and  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$  are constant parameters. The values of the constant parameters corresponding to the condition of "best fit" can be obtained by means of a statistical procedure called regression analysis (5).

### 1.3 Discussion of Results

All kinds of strength-size variation are shown in the plots. In the harder rocks (St. Cloud gray granodiorite and Valders limestone in some cases) there appears to be a definite correlation between size and strength. Some agree with Koifman's prediction. The apparent absence of definite pattern of the size-strength relationship in the other plots could be attributed to one or a combination of the following:

- (a) The size differences between the specimens were not large enough.
- (b) Not enough samples were tested for certain sizes particularly in strain-controlled tests.
- (c) Valders limestone and dacite are not nearly homogenous and isotropic as first thought. The tensile strength was therefore, affected more by the position and orientation

DACITE  
STRAIN CONTROL

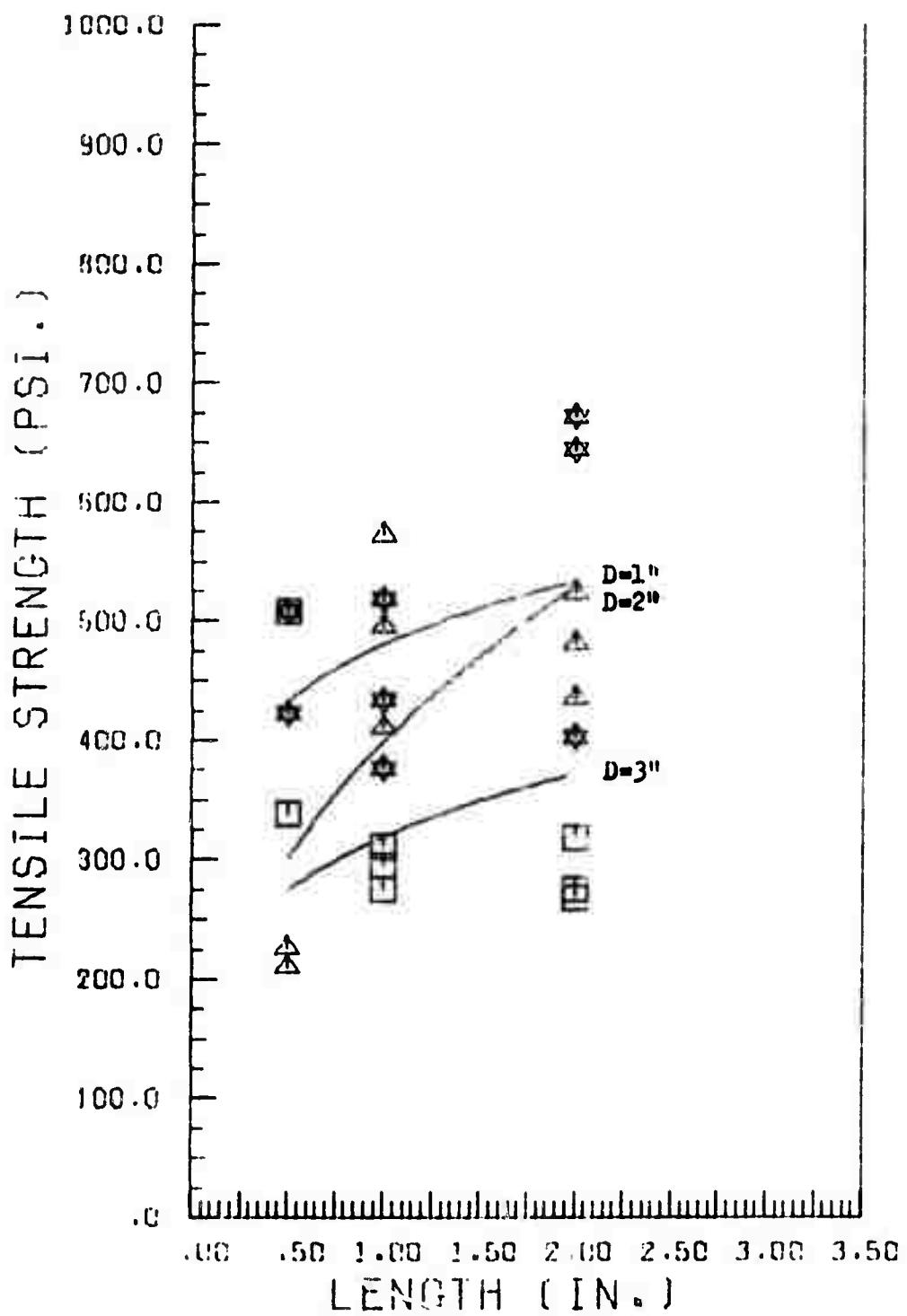


Figure 1.3

DACITE  
STRAIN CONTROL

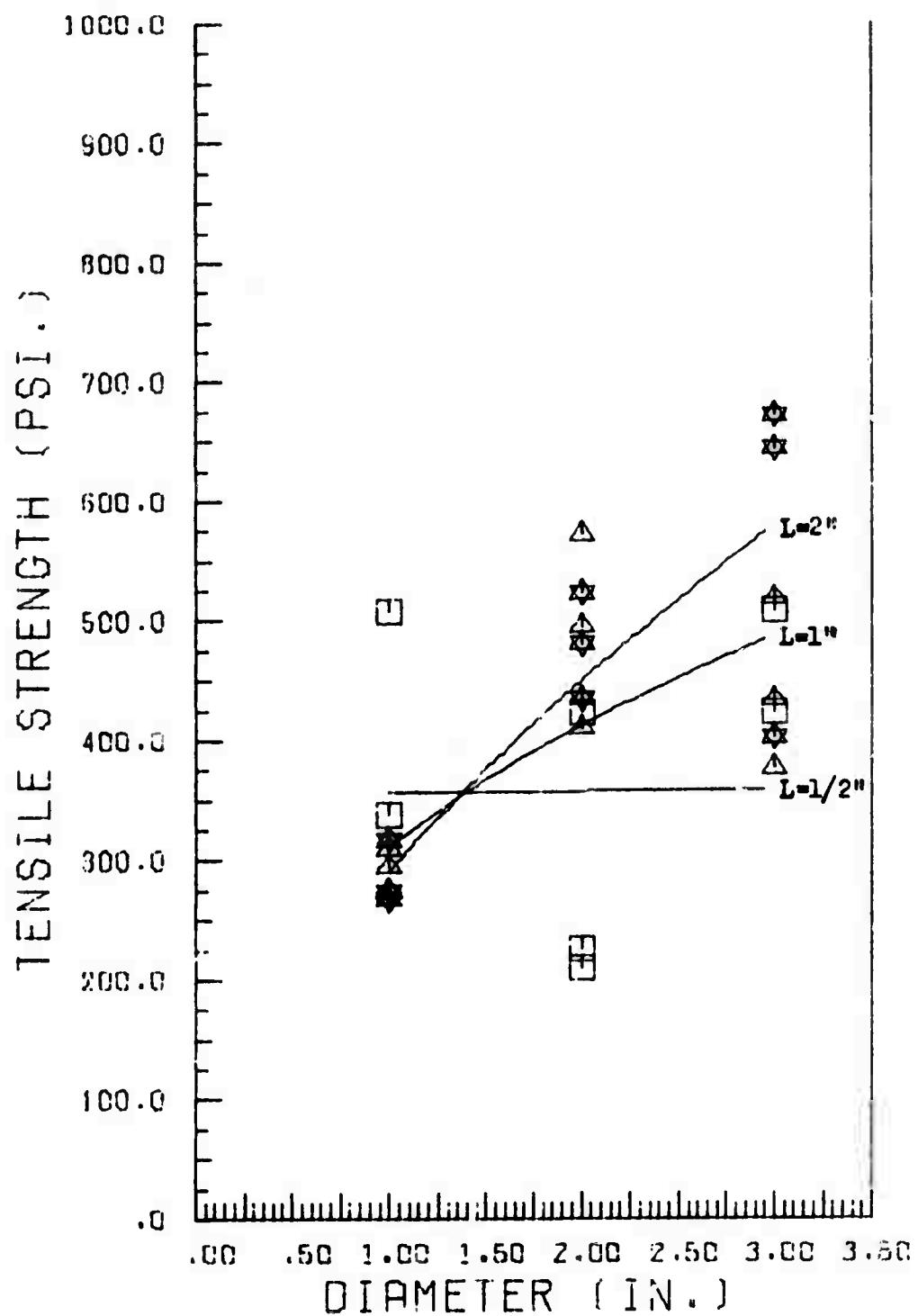


Figure 1.4

DACITE  
STRESS CONTROL

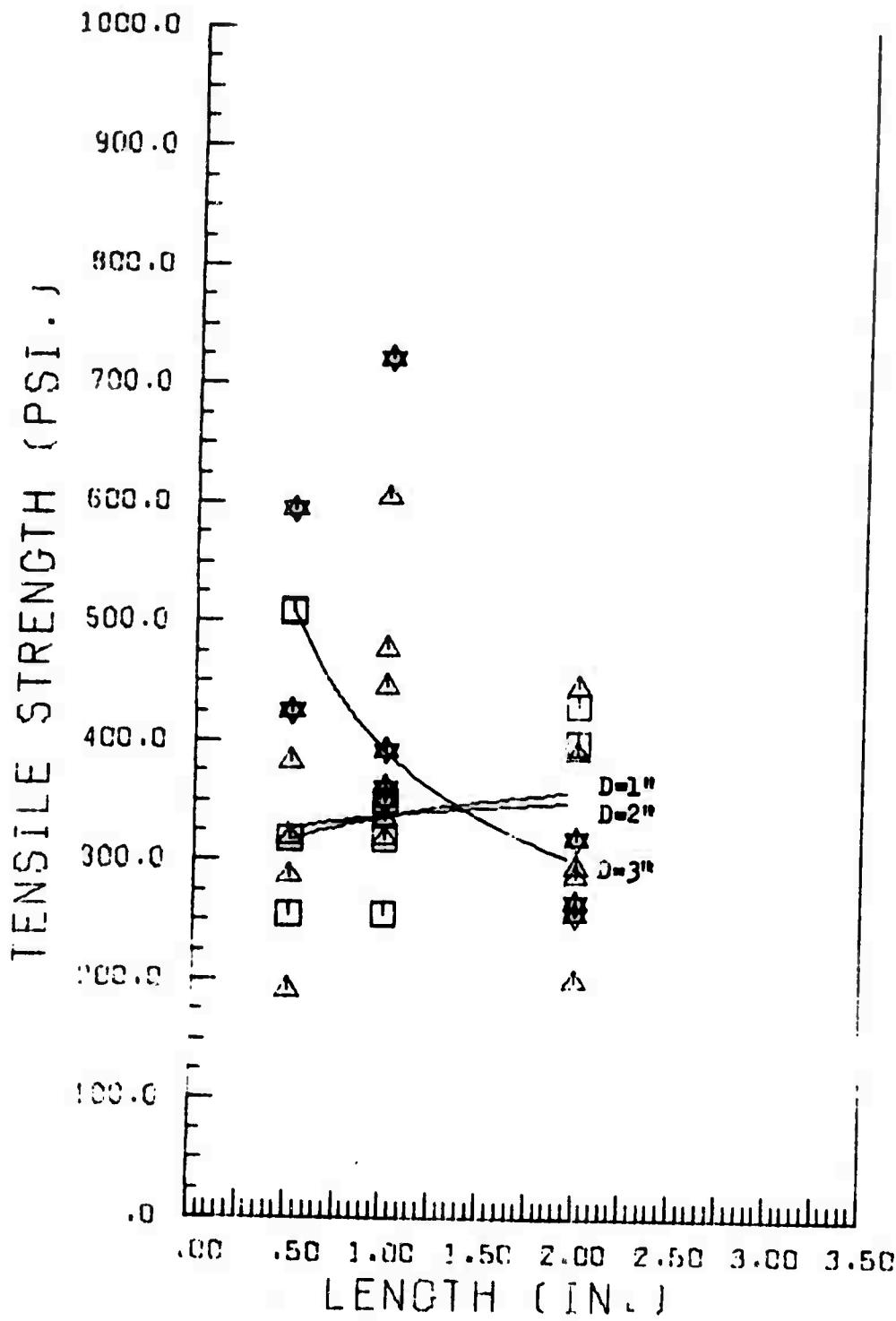


Figure 1.5

DACITE  
STRESS CONTROL

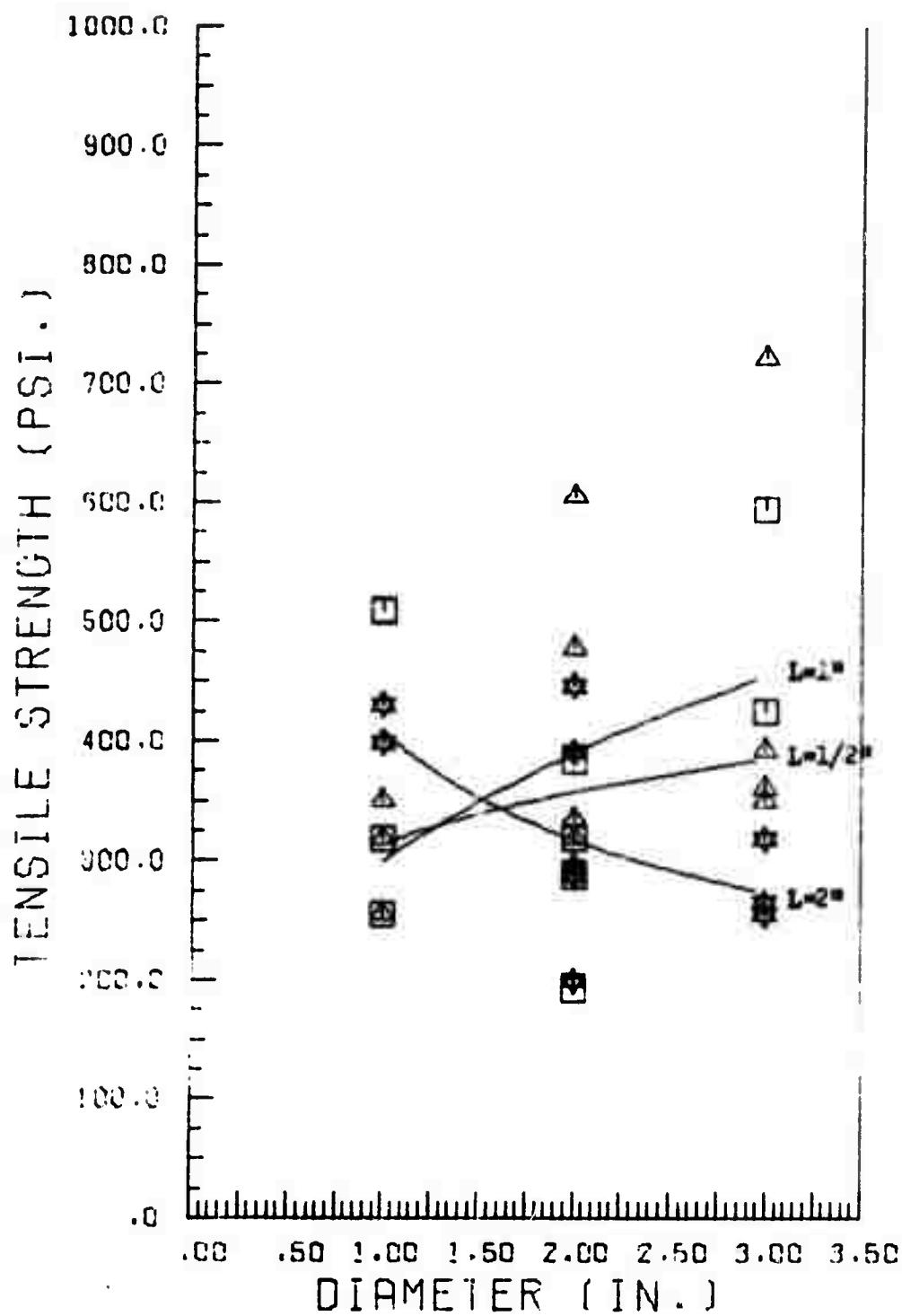


Figure 1.6

# VALDERS LIMESTONE STRAIN CONTROL

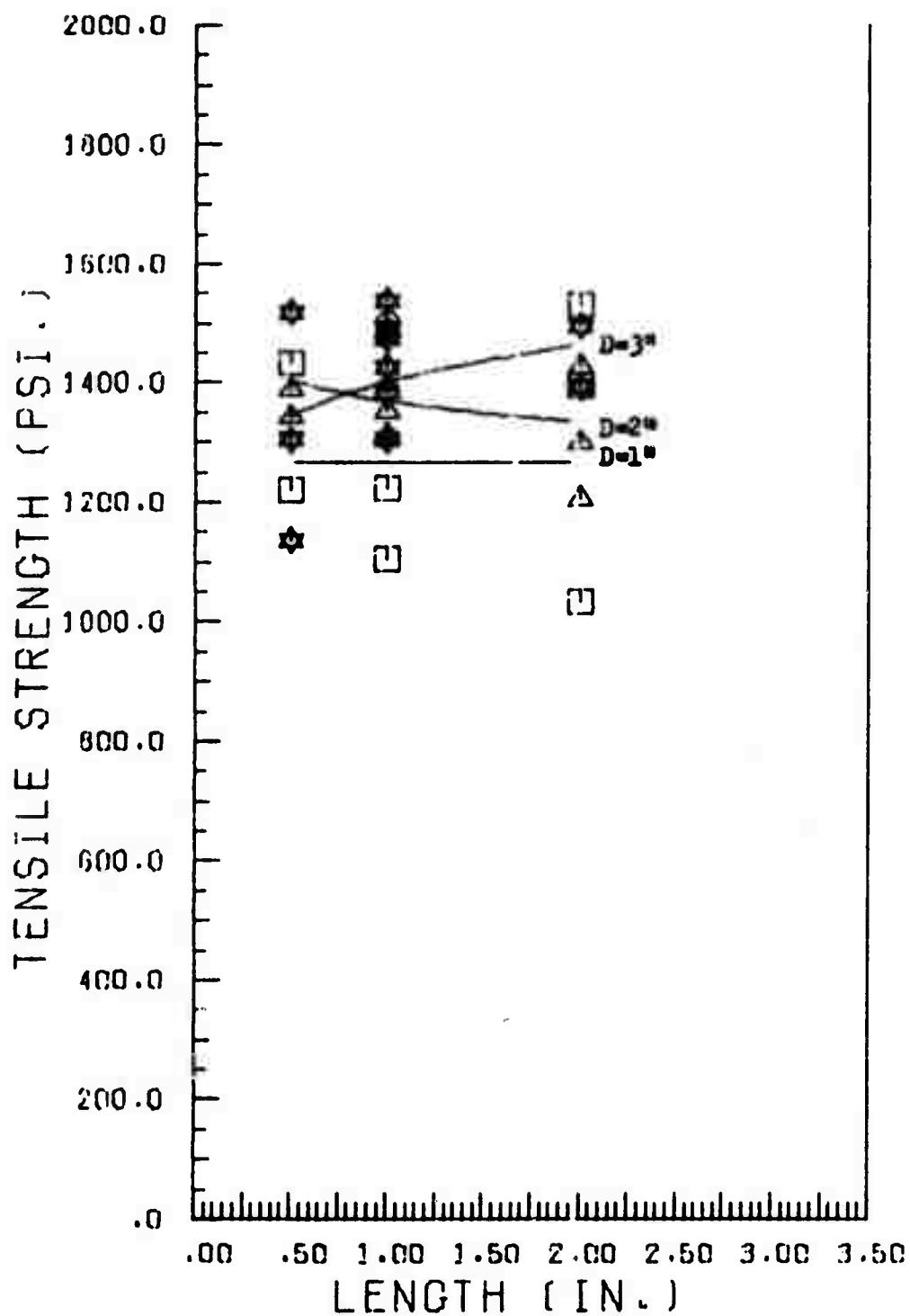


Figure 1.7

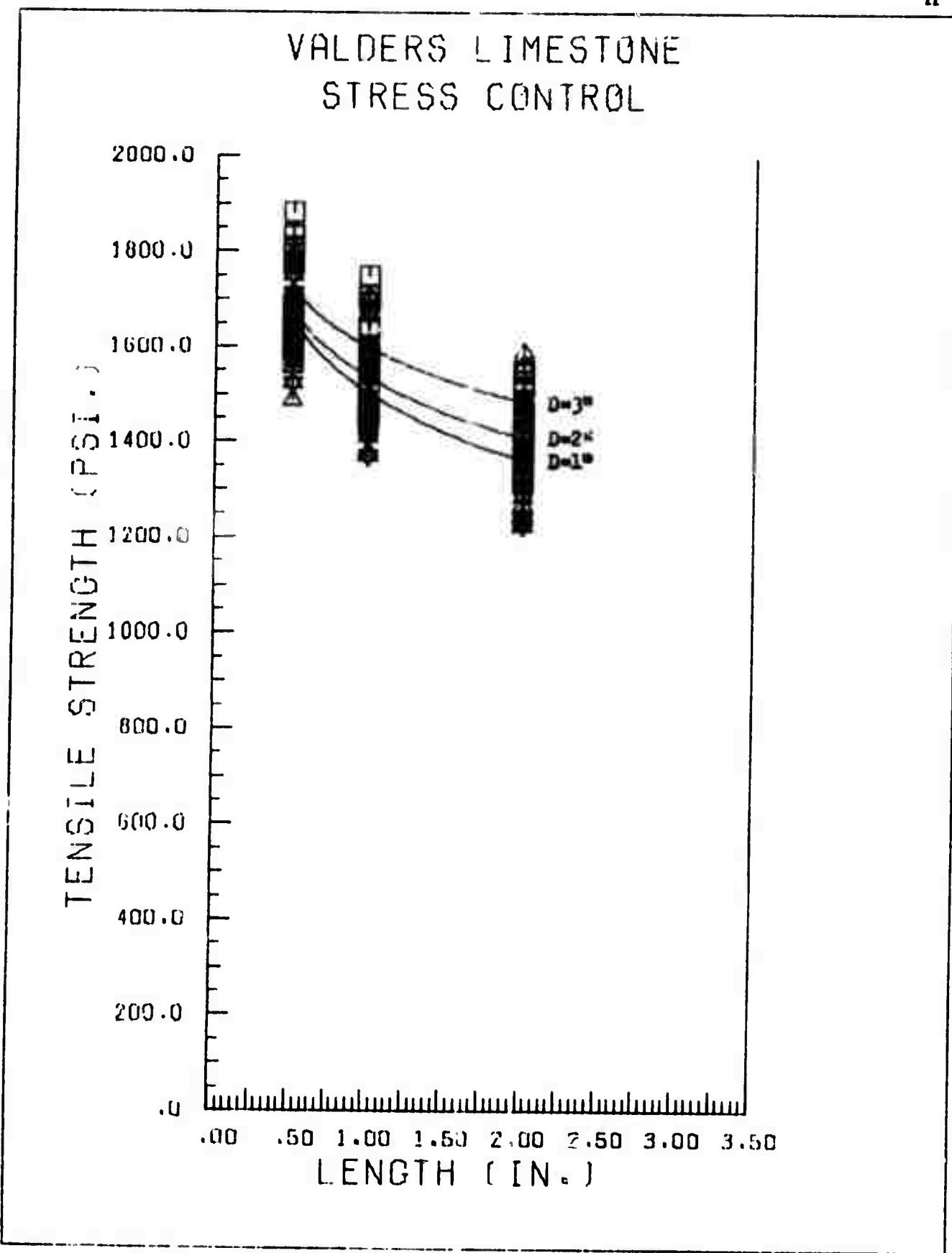


Figure 1.8

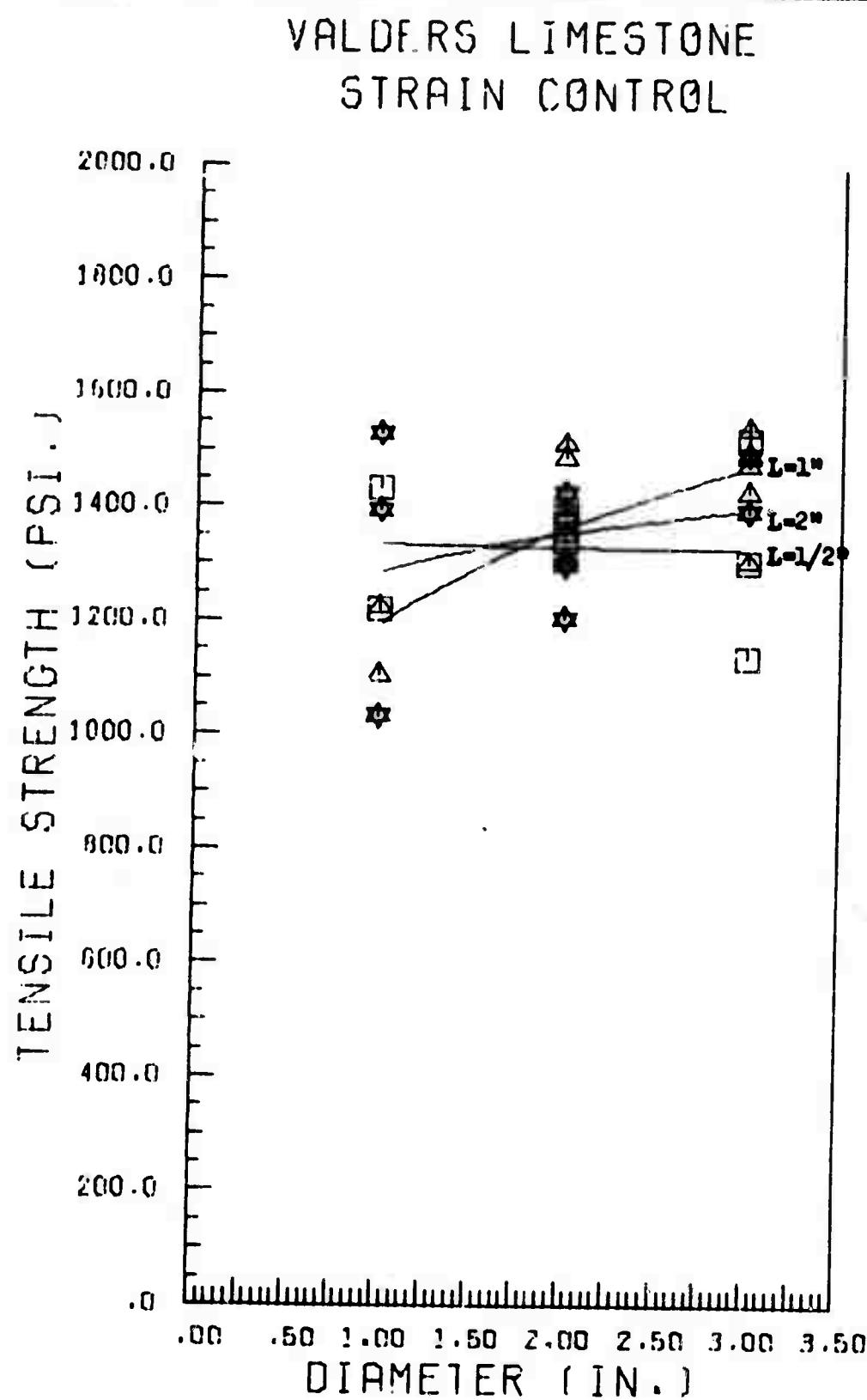


Figure 1.9

WILDFERS LIMESTONE  
STRESS CONTROL

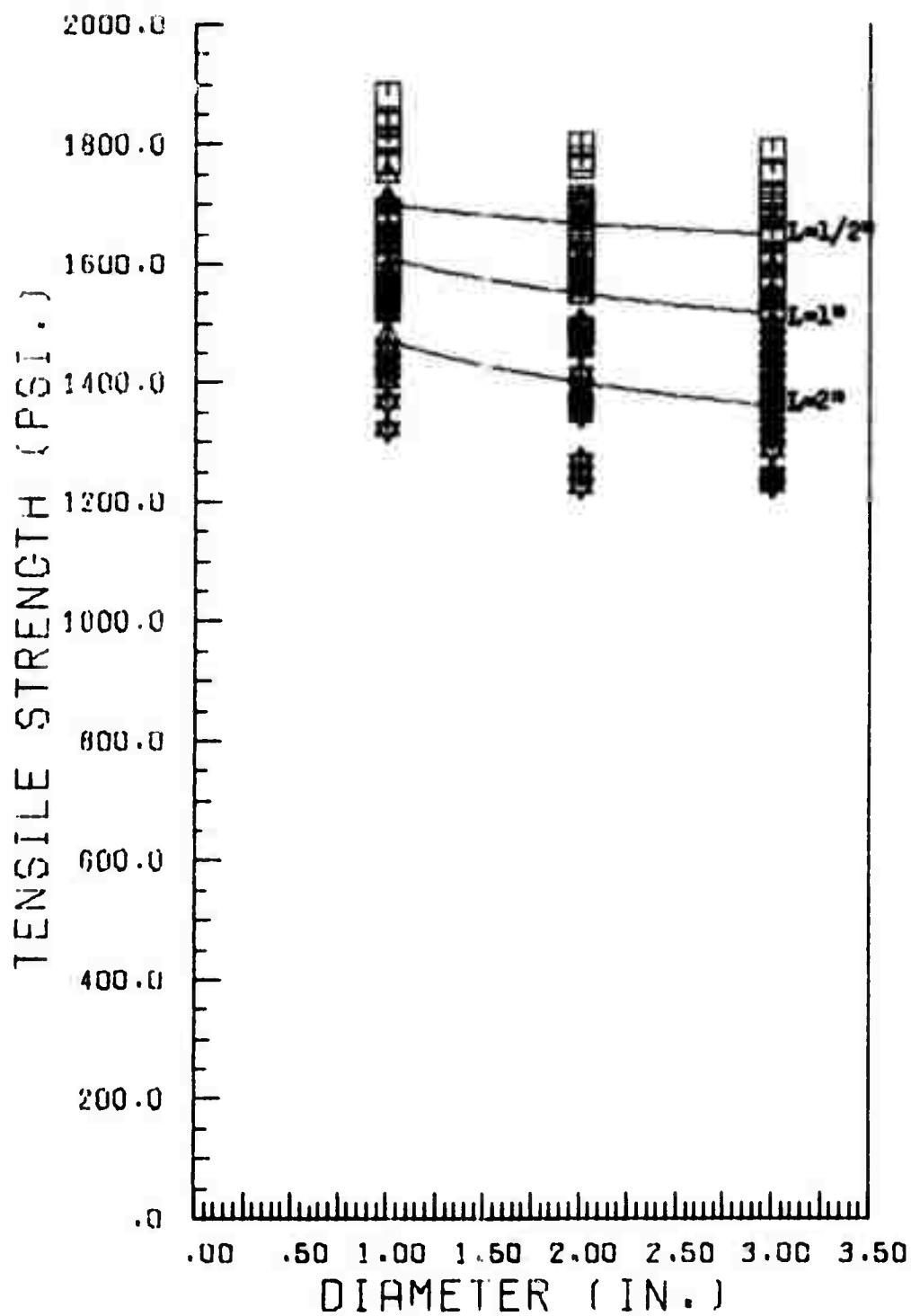


Figure 1.10

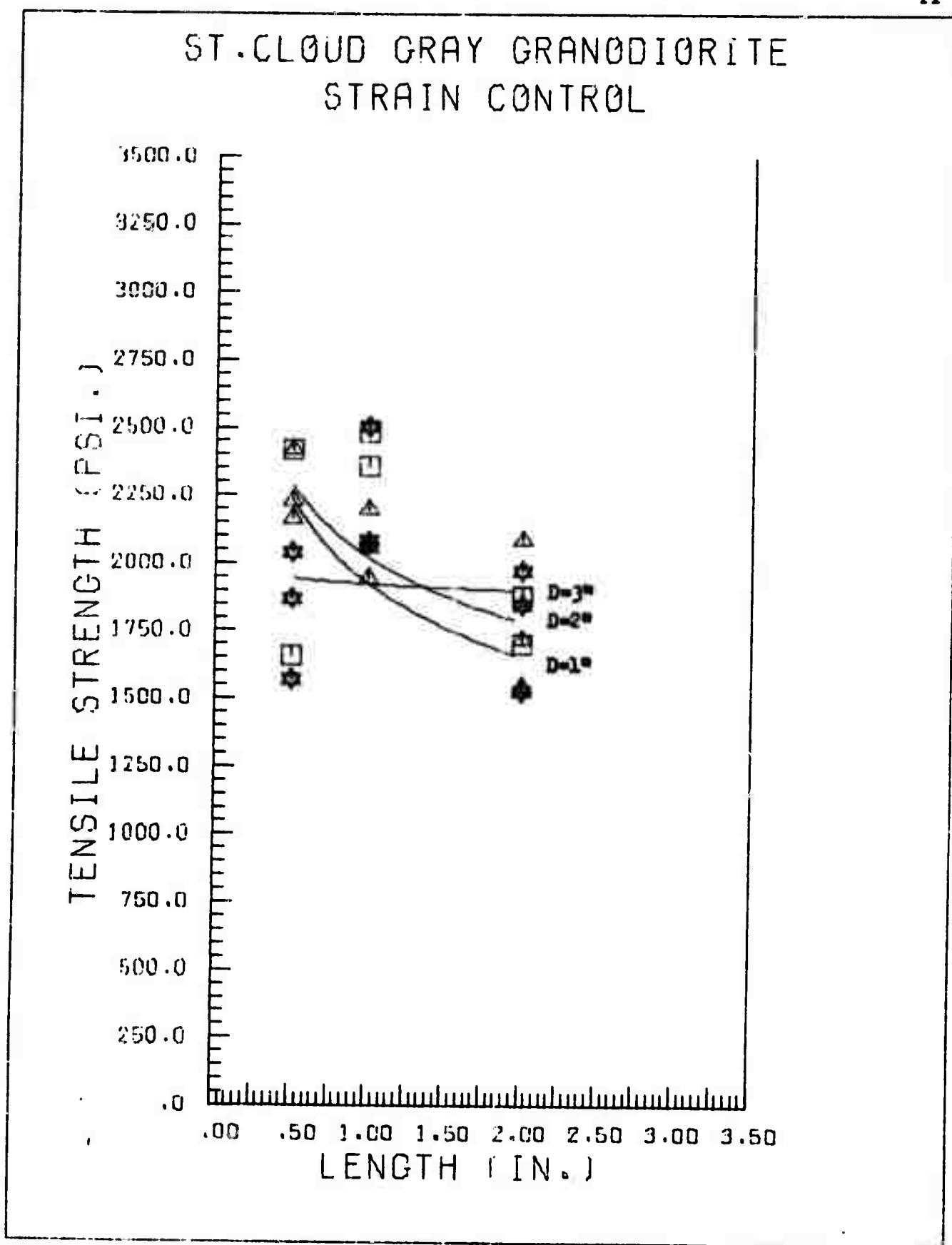


Figure 1.11

ST.CLOUD GRAY GRANODIORITE  
STRESS CONTROL

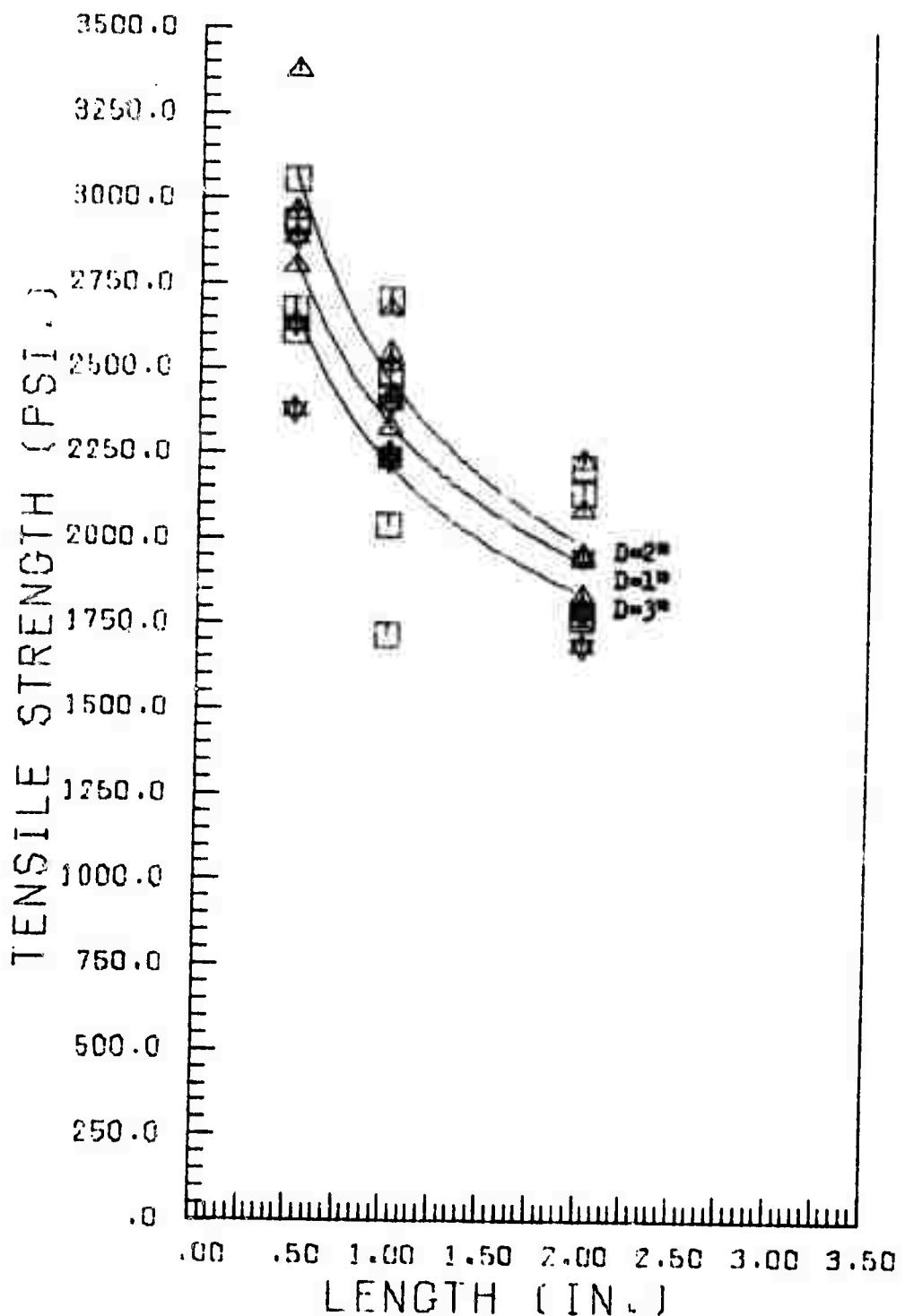


Figure 1.12

ST.CLOUD GRAY GRANODIORITE  
STRAIN CONTROL

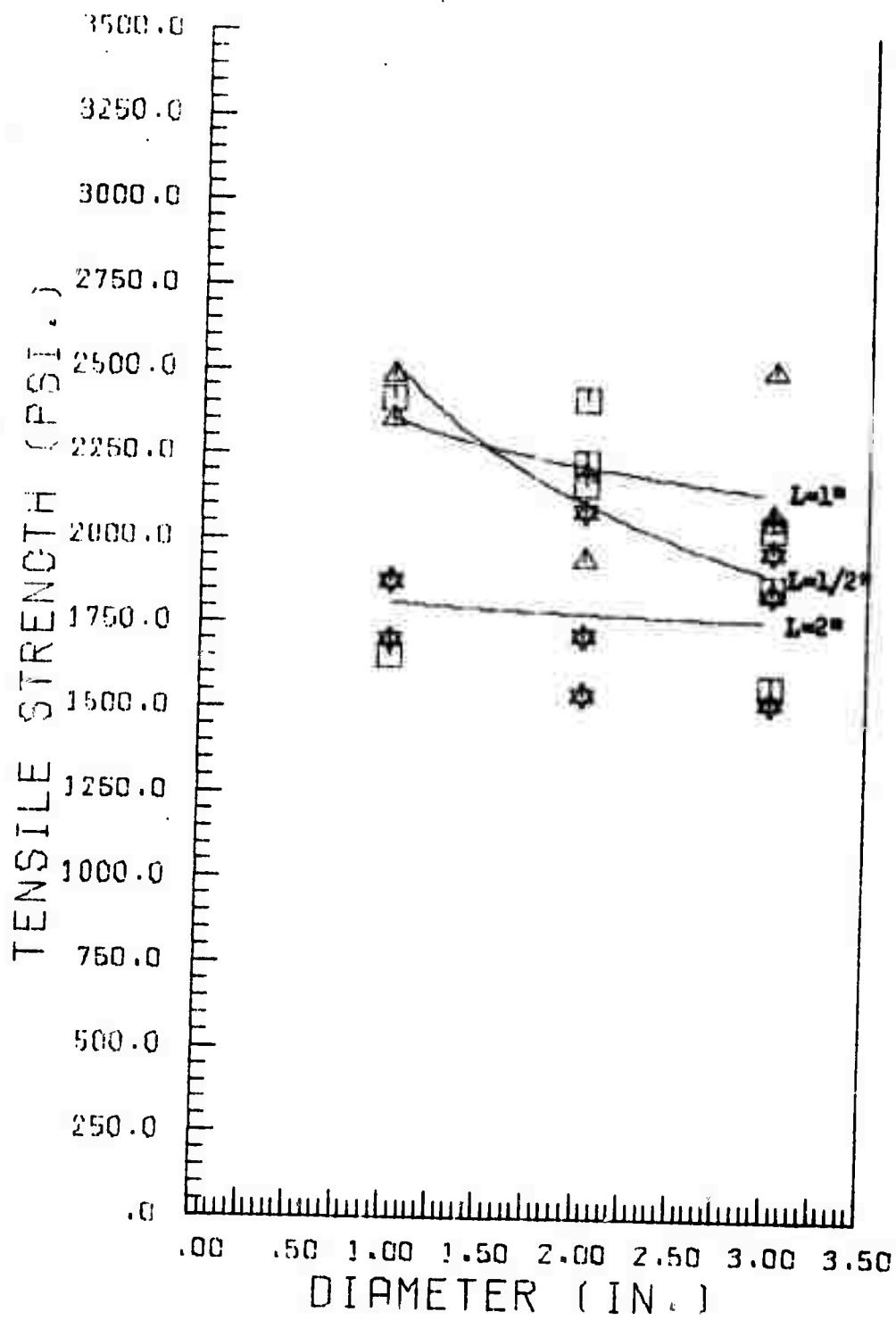


Figure 1.13

ST.CLOUD GRAY GRANODIORITE  
STRESS CONTROL

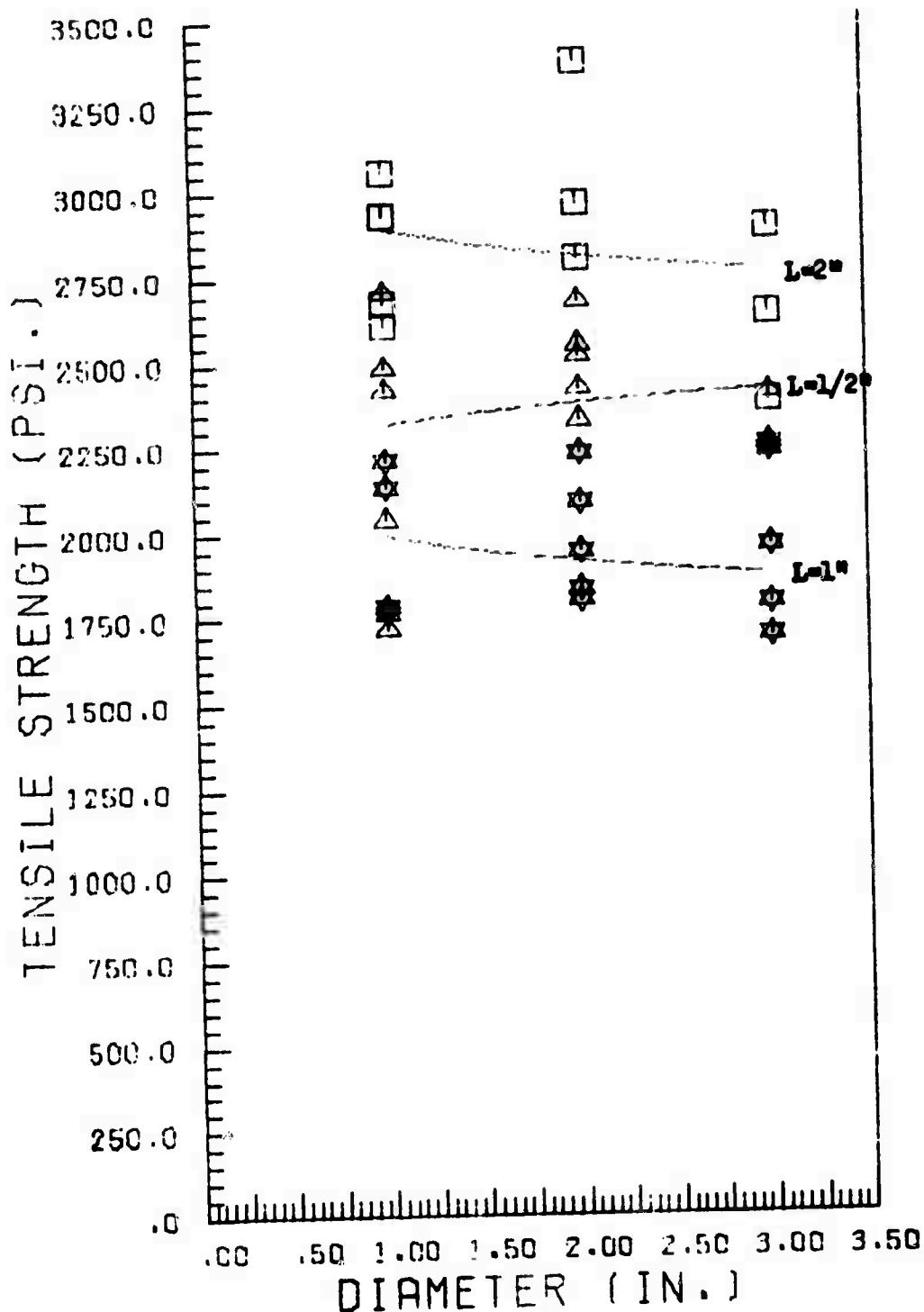


Figure 1.14

of the internal structural defects with respect to the loaded diametral plane than by the number and extent of said defects. This also explains the scattering of the data points.

#### 1.4 Conclusions

There is definitely a size effect on tensile strength. In Brazilian tests, the size effect will be governed mainly by the position and orientation of the internal defects relative to the loaded diametral plane rather than by the extent and number of the defects. For this reason, the Brazilian test is not a good basis for studying size-tensile strength dependence unless care is taken to consistently load the specimens along the same diametral plane. The splitting test described by Koifman (6) appears to be a better alternative.

The size difference between the specimens used in this study was not large enough to predict a definite pattern of size-strength relationship especially in the case of the soft rocks. Future tests should include larger specimens.

## REFERENCES

1. Heins, R. W., et al, "Aspects of Mechanical Behavior of Rock Under Static and Cyclic Loading (Part A: Mechanic Behavior of Rock Under Static Loading)", First Annual Progress Report, Engineering Experiment Station, University of Wisconsin, Madison, Wisconsin, March 1972.
2. Brown, E. T., "Strength-Size Effect in Rock Material", Progress Report No. 24, University of Minnesota Mineral Resources Center, April 1971.
3. Koifman, M. I., "The Size Factor in Rock Pressure Investigations", Mechanical Properties of Rocks by M. M. Protod'yakonov, M. I. Koifman, and others, Moscow Acad. Sci., USSR (translated by Israel Prog. Sci. Trans., Jerusalen), 1963.
4. Jaeger, J. C. and N. G. W. Cook, Fundamentals of Rock Mechanics, Methuen & Co., Ltd., London, 1969.
5. Draper, N. R. and H. Smith, Applied Regression Analysis, John Wiley and Sons, Inc., New York, 1966.
6. Koifman, M. I., "Quick Combined Method of Determining Mechanical Properties of Rocks", Mechanical Properties of Rocks by M. M. Protod'yakonov, M. I. Koifman, and others, Moscow Acad. Sci., USSR (translated by Israel Prog. Sci. Trans., Jerusalem), 1963.

## CHAPTER 2

### FINITE ELEMENT SIMULATION OF BRAZILIAN TEST

#### 2. 1 Introduction

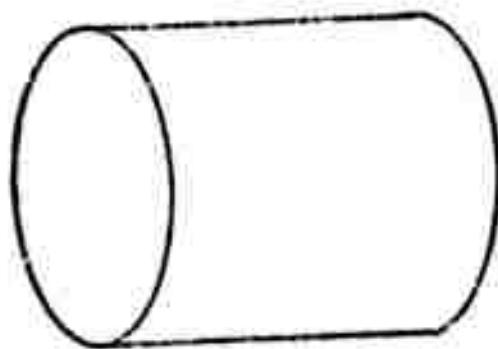
A theoretical version of the Brazilian test employing the finite element technique is described in this chapter. In connection with the proposed method, development of a highly efficient computer program which will be based on both three-dimensional and two-dimensional elements and which will account for material anisotropy as well as nonhomogeneity is now underway. A two-dimensional program which takes into account nonhomogeneity but not anisotropy has already been developed and will be described in the next chapter.

In the treatment of nonhomogeneous problems, the basic idea suggested in the first annual report will be used; that is, elastic properties will be assigned randomly to each element by means of a random number generating routine. Other failure criteria not touched in the first annual report will be investigated. It has been experimentally demonstrated (1) that although rock is essentially a brittle material, it attains an unusually high degree of ductility when subjected to high confining pressures. This phenomenon, however, is vaguely defined in literature and will, therefore, be taken into account only approximately when considering post-failure behavior of elements.

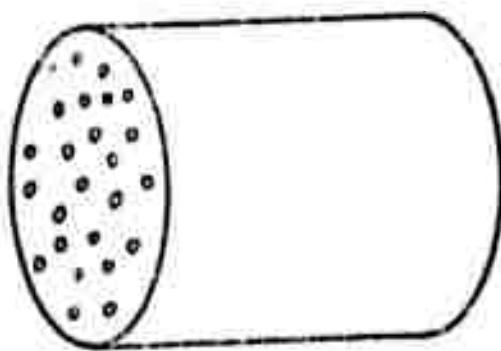
In writing this report, it is assumed that the reader is familiar with the basic principles of matrix algebra and the finite element method. No attempt will be made to rederive equations which have already been derived in previous publications.

#### 2. 2 Formulation of Essential Matrices for Three-Dimensional Elements

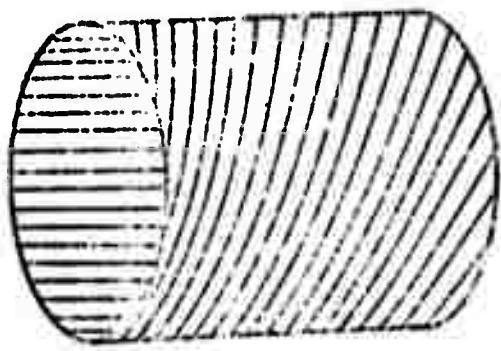
In the finite element method of analysis, the whole structural system is idealized as an assemblage of elements which are connected to one another only at a discrete number of nodal points. The nodal displacements are the



(a) Homogeneous and isotropic



(b) Nonhomogeneous



(c) Homogeneous and anisotropic

FIGURE 2.1 Material Properties Considered in Analysis

basic unknown quantities of the method upon which the displacement pattern and therefore the stresses within the boundaries of the element depend. To facilitate the calculation of the element stiffness properties, a set of displacement functions, usually polynomials, are assumed. These functions uniquely define the deformations allowed within an element in terms of the nodal displacements.

In this study, the Brazilian test specimen is divided by imaginary annular surfaces and radial planes into elements of the kind shown in Fig. 2.2. The corner points of the elements are designated as the nodal points. Eight-term linear polynomials are used to represent the radial ( $u$ ), circumferential ( $v$ ), and axial ( $w$ ) displacements within an element. Thus

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} 1 & r' & \theta' & r'\theta' & z' & r'z' & \theta'z' & r'\theta'z' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & r' & \theta' & r'\theta' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{24} \end{Bmatrix} \quad (2.1)$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ z' & r'z' & \theta'z' & r'\theta'z' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & r' & \theta' & r'\theta' & z' & r'z' & \theta'z' & r'\theta'z' \end{bmatrix}$$

where  $a_1, a_2, \dots, a_{24}$  are the constant coefficients of the polynomials and  $(r', \theta', z')$  are local dimensionless cylindrical coordinates which range in value from -1 to +1 within an element. The global cylindrical coordinates  $(r, \theta, z)$  are related to the local coordinates as follows:

$$\begin{aligned} r &= r_o + r_s r' \\ \theta &= \theta_o + \theta_s \theta' \\ z &= z_o + z_s z' \end{aligned} \quad (2.2)$$

where  $(r_o, \theta_o, z_o)$  are the global coordinates of the origin of the local coordinate axes and  $2r_s, 2\theta_s$  and  $2z_s$  are the side dimensions of the element (see Fig. 2.3).

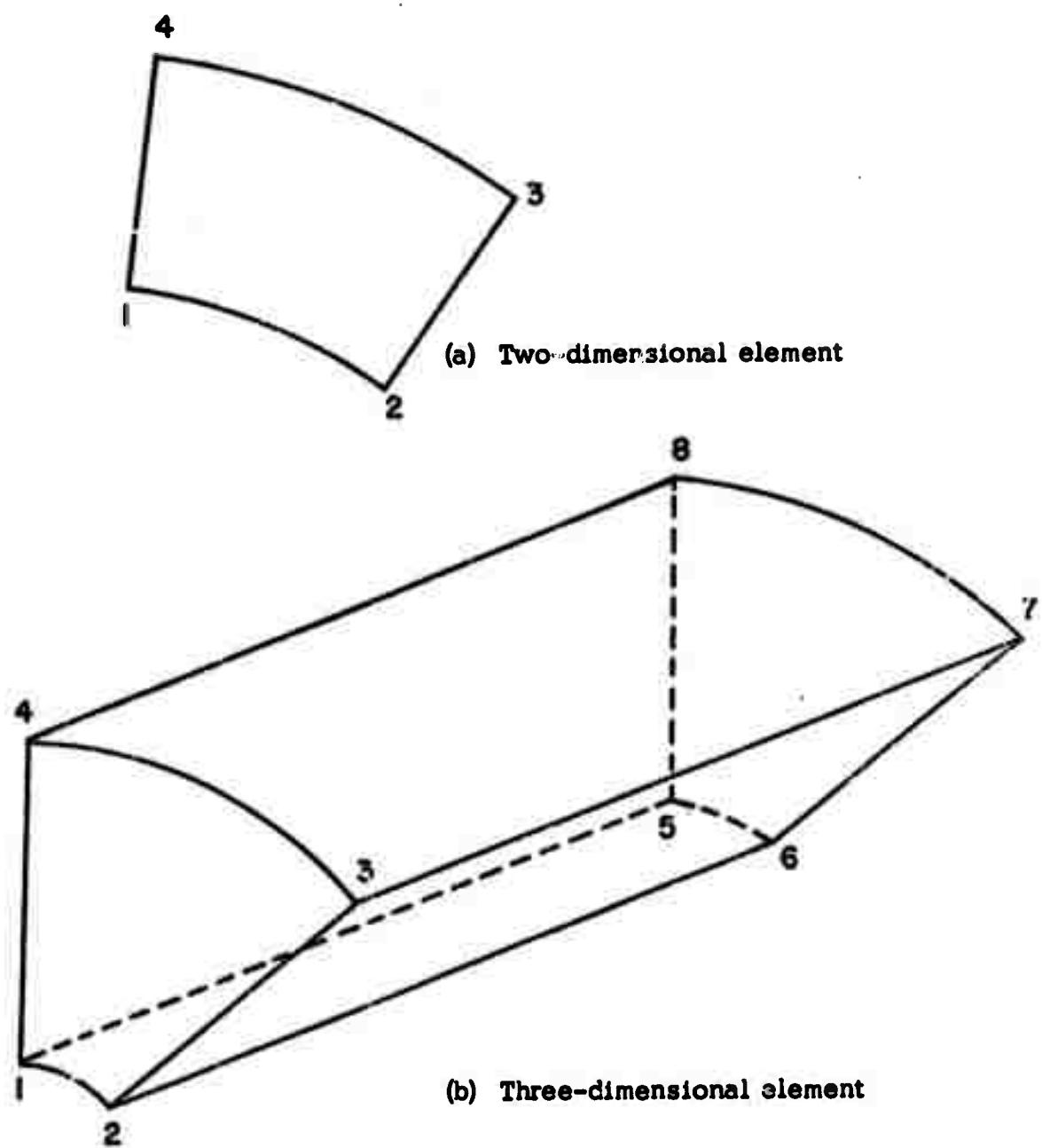


FIGURE 2.2 Element Models With Local Node Numbers Indicated

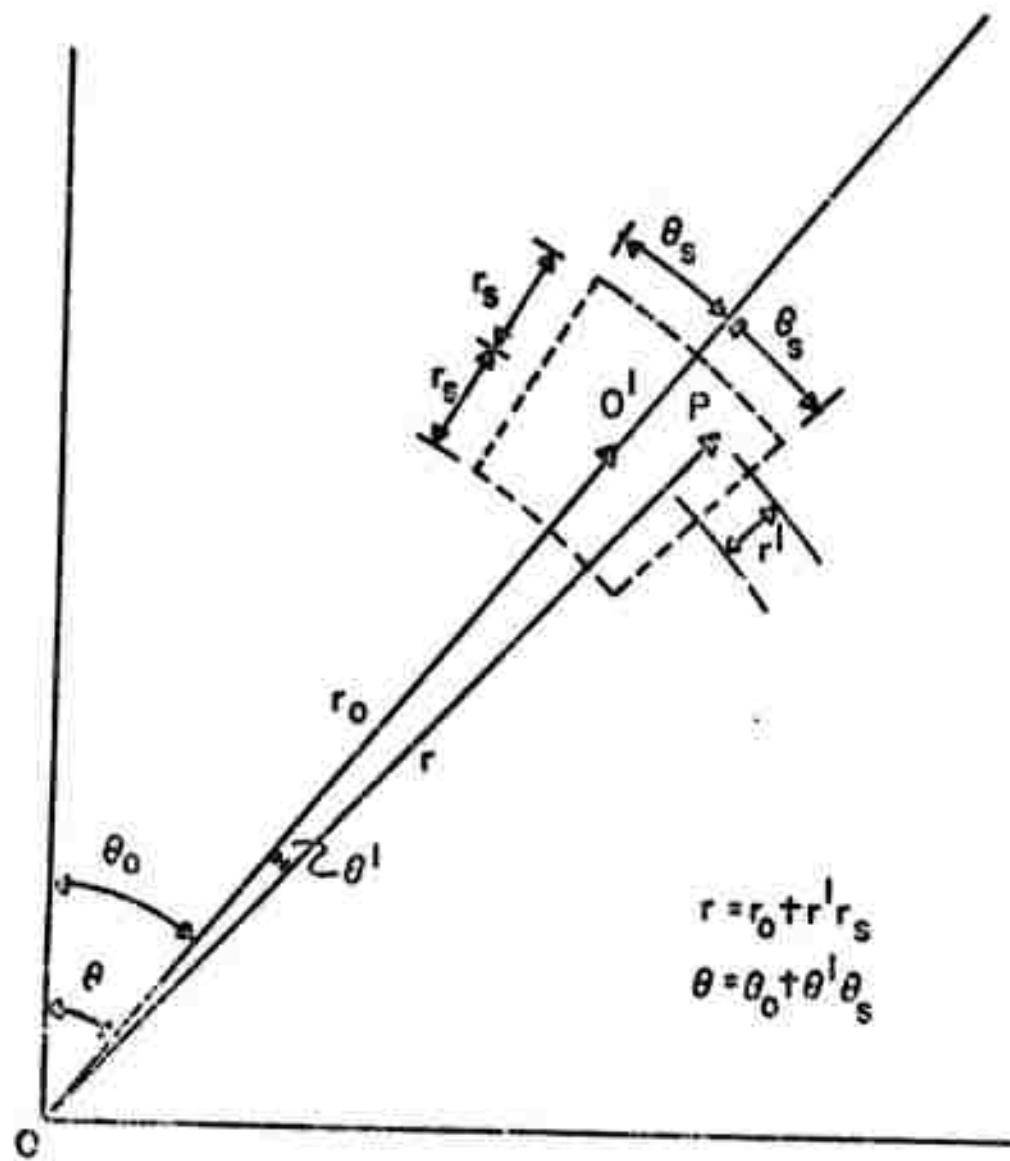


FIGURE 2.3 Relationship Between Global and Local Coordinates

To obtain the displacements at any node  $i$ , one merely substitutes into eq. (2. 1) the appropriate nodal coordinates, that is,

$$\begin{aligned} u_i &= a_1 + a_2 r'_i + a_3 \theta'_i + a_4 r' \theta'_i + \dots \\ v_i &= a_9 + a_{10} r'_i + a_{11} \theta'_i + a_{12} r' \theta'_i + \dots \\ w_i &= a_{17} + a_{18} r'_i + a_{19} \theta'_i + a_{20} r' \theta'_i + \dots \end{aligned} \quad (2. 3)$$

where  $r'_i$ ,  $\theta'_i$ ,  $z'_i$  are equal to +1 or -1. Thus, if  $(x)^*$  denotes the nodal displacement vector  $\{u_1 \ v_1 \ w_1 \ u_2 \ v_2 \ \dots \ v_8 \ w_8\}$  and  $\{a\}$  the constant coefficient array, then

$$(x) = [c] \{a\} \quad (2. 4)$$

The matrix  $[c]$  is shown in Table 2. 1. It can easily be verified that

$$[c]^{-1} = \frac{1}{8} [c]^T \quad (2. 5)$$

From the equations for the components of strain at a point,

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \epsilon_z \\ \gamma_{r\theta} \\ \gamma_{rz} \\ \gamma_{\theta z} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{u}{r} + \frac{\partial v}{r \partial \theta} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{r \partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{r \partial \theta} \end{Bmatrix} \quad (2. 6)$$

one obtains, with the aid of eqs. (2. 1) and (2. 2), the relationship

$$\{\epsilon\} = [q] \{a\} \quad (2. 7)$$

where the elements of  $[q]$  are listed in Table 2. 2.

The stress components are related to the strain components by the elasticity matrix  $[D]$ , that is

$$\{\delta\} = [D] \{\epsilon\} \quad (2. 8)$$

where, for the isotropic case

---

\*The symbol  $\{ \}$  denotes column matrices while  $[ ]$  denotes all other matrices.

$$[D] = \begin{bmatrix} \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \text{Symmetric} & & & G & 0 & 0 \\ & & & & G & 0 \\ & & & & & G \end{bmatrix}$$

$E$  = modulus of elasticity

$\nu$  = Poisson's ratio

$G$  = shear modulus =  $\frac{E}{2(1+\nu)}$

and

$(\sigma) = (\sigma_r \ \sigma_\theta \ \sigma_z \ \tau_{r\theta} \ \tau_{rz} \ \tau_{\theta z})$  (see Fig. 2.5)

The formulation of  $[D]$  for the case where the material is anisotropic requires added consideration. Let 1, 2, 3 be the axes of anisotropy (assumed mutually perpendicular) and  $\alpha_1, \alpha_2, \alpha_3$  the angles which define the orientation of these axes. To understand more clearly the significance of the parameters  $\alpha_1, \alpha_2, \alpha_3$ , a step-by-step rotation of the axes 1, 2, 3 to their actual positions is illustrated in Fig. 2.4. Let axes 2, 3 lie in the plane ABCD. In Fig. 2.4a, axis 3 and plane ABCD are initially positioned parallel to the Z-axis of the element,  $\alpha_1$  being the angle which plane ABCD makes with the radial plane passing through the central point of the element. In Fig. 2.4b, plane ABCD is rotated an angle  $\alpha_2$  about axis 2 to A'B'C'D'. Figure 2.4c shows the actual orientation of axes 1, 2, 3 arrived at by rotating axes 2, 3 an angle  $\alpha_3$  in their own plane (A'B'C'D'). It should be noted that if material properties do not vary in the plane of 2, 3, then  $\alpha_3$  can be arbitrarily set to any value, say zero.

The stress-strain relations for general anisotropy are given in the theory of elasticity as

$$\begin{aligned}
 \epsilon_1 &= \frac{\sigma_1}{E_1} - \frac{\nu_{12}\sigma_2}{E_2} - \frac{\nu_{13}\sigma_3}{E_3} \\
 \epsilon_2 &= -\frac{\nu_{12}\sigma_1}{E_2} + \frac{\sigma_2}{E_2} - \frac{\nu_{23}\sigma_3}{E_3} \\
 \epsilon_3 &= -\frac{\nu_{13}\sigma_1}{E_3} - \frac{\nu_{23}\sigma_2}{E_3} + \frac{\sigma_3}{E_3}
 \end{aligned} \tag{2. 9}$$

$$\gamma_{12} = \frac{1}{G_{12}} \tau_{12}$$

$$\gamma_{13} = \frac{1}{G_{13}} \tau_{13}$$

$$\gamma_{23} = \frac{1}{G_{23}} \tau_{23}$$

where the subscripts refer to the directions of anisotropy defined in the preceding paragraph. Solving for the stresses and writing the resulting equations in matrix form, one obtains

$$(\sigma') = [D'] (\epsilon') \tag{2. 10}$$

where

$$(\sigma') = \{\sigma_1 \ \sigma_2 \ \sigma_3 \ \sigma_{12} \ \sigma_{13} \ \sigma_{23}\}$$

$$(\epsilon') = \{\epsilon_1 \ \epsilon_2 \ \epsilon_3 \ \gamma_{12} \ \gamma_{13} \ \gamma_{23}\}$$

and

$$[D']^{-1} = \left[ \begin{array}{ccc|c}
 \frac{1}{E_1} - \frac{\nu_{12}}{E_2} - \frac{\nu_{13}}{E_3} & 0 & 0 & 0 \\
 \frac{1}{E_2} - \frac{\nu_{23}}{E_3} & 0 & 0 & 0 \\
 \frac{1}{E_3} & 0 & 0 & 0 \\
 \hline
 & \frac{1}{G_{12}} & 0 & 0 \\
 & & \frac{1}{G_{13}} & 0 \\
 & & & \frac{1}{G_{23}}
 \end{array} \right] \text{Symmetric}$$

1	-1	-1	1	-1	1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	-1	-1	1	-1	1	1	-1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	-1	1	1	-1
1	-1	1	-1	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	-1	1	-1	-1	1	-1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	-1	1	-1	1
1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	-1	-1	-1	-1
1	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	-1	-1	-1	1	1
1	-1	-1	1	1	-1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	-1	-1	1	1	-1	-1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	-1	-1	-1	1	-1	1
1	-1	1	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	-1	1	-1	1	-1	1	-1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	1	-1	1	-1
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
1	1	-1	1	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	-1	-1	1	1	-1	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

Table 2.1 The Matrix [c]

$$\left[ \begin{array}{cccccc} 0 & \frac{1}{z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \end{array} \right] = \left[ \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{z} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{z} \\ \end{array} \right]$$

Table 2.2 The Matrix [q]

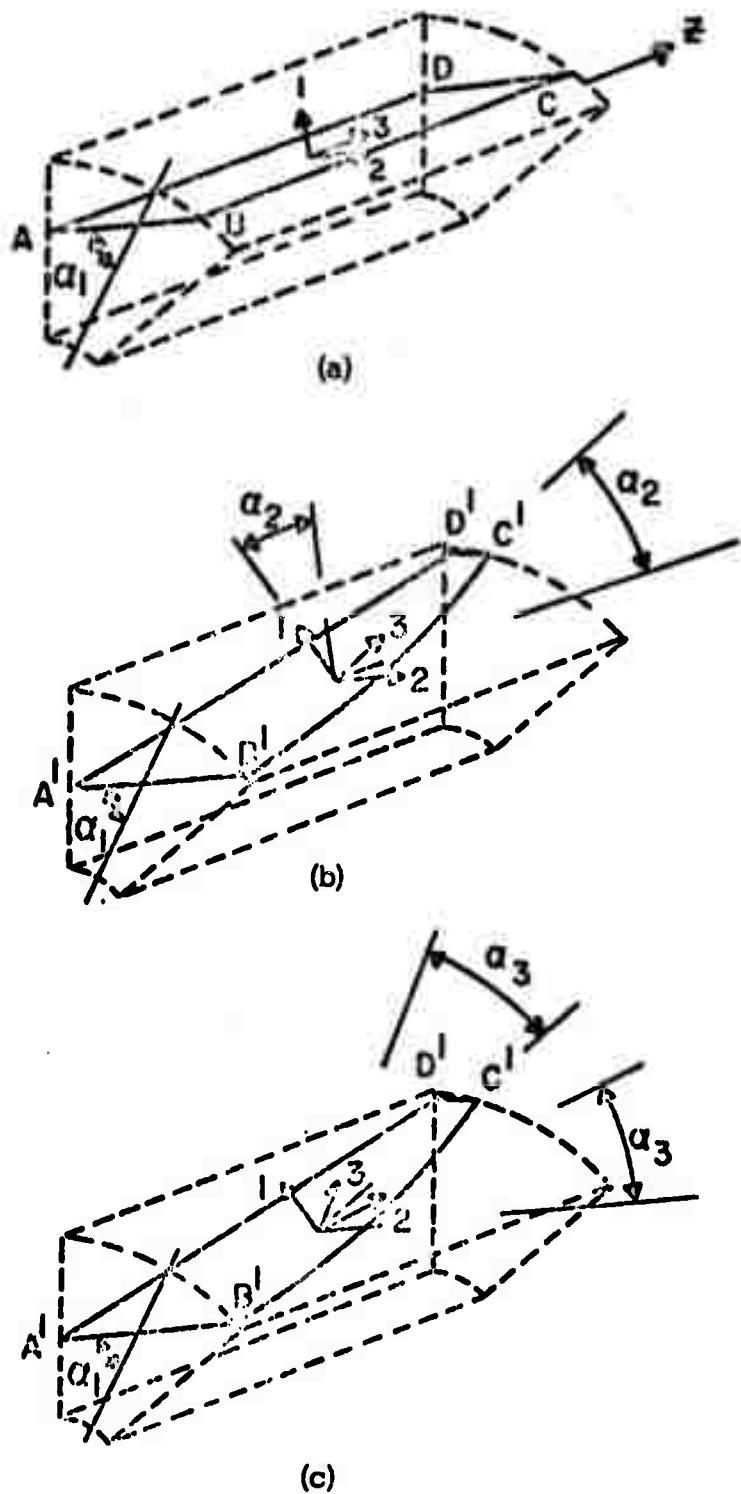
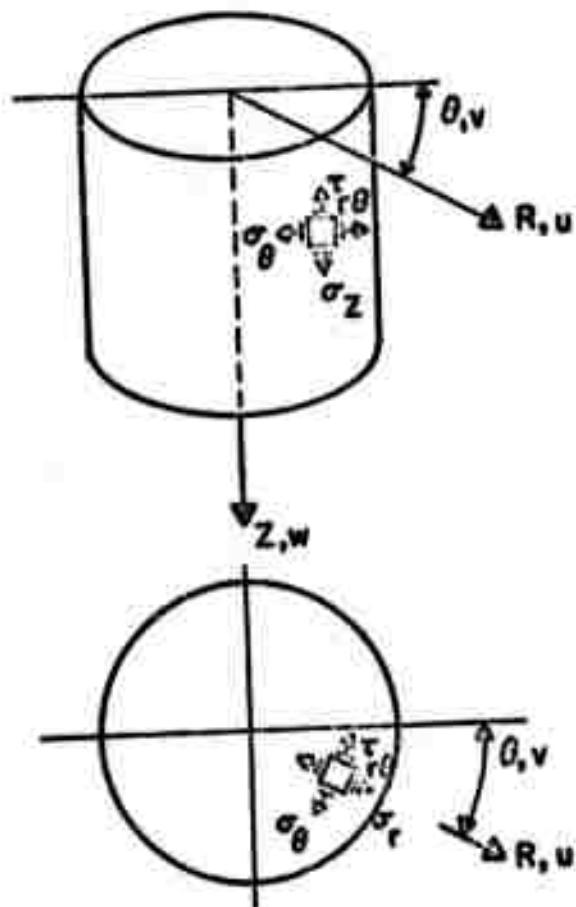


FIGURE 2.4 Step-by-step Orientation of Axes of Anisotropy



**FIGURE 2.5 Positive Direction of Stresses and Displacements**

An equation connecting the stress vectors  $\{\sigma\}$  and  $\{\sigma'\}$  can be obtained from consideration of equilibrium of the tetrahedron shown in Fig. 2.6. Let, for example, the stresses  $\sigma_\theta$ ,  $\tau_{r\theta}$  and  $\tau_{\theta z}$  act on the inclined plane ABC. Furthermore, let  $(t_{21}, t_{22}, t_{23})$ ,  $(t_{11}, t_{12}, t_{13})$ , and  $(t_{31}, t_{32}, t_{33})$  be the direction cosines of these stresses relative to the axes 1, 2, 3. It can be shown that the components of stress acting on the plane ABC and parallel to the coordinate axes 1, 2, 3 are

$$\begin{aligned} R_1 &= \sigma_1 t_{21} + \tau_{12} t_{22} + \tau_{13} t_{23} \\ R_2 &= \tau_{12} t_{21} + \sigma_2 t_{22} + \tau_{23} t_{23} \\ R_3 &= \tau_{13} t_{21} + \tau_{23} t_{22} + \sigma_3 t_{23} \end{aligned} \quad (2.11)$$

The stress components  $\sigma_\theta$ ,  $\tau_{r\theta}$  and  $\tau_{\theta z}$  can then be calculated to be

$$\begin{aligned} \sigma_\theta &= R_1 t_{21} + R_2 t_{22} + R_3 t_{23} \\ \tau_{r\theta} &= R_1 t_{11} + R_2 t_{12} + R_3 t_{13} \\ \tau_{\theta z} &= R_1 t_{31} + R_2 t_{32} + R_3 t_{33} \end{aligned} \quad (2.12)$$

Similar expressions can be derived for  $\sigma_r$ ,  $\sigma_z$  and  $\tau_{rz}$  resulting in the general relationship

$$\{\sigma\} = [T] \{\sigma'\} \quad (2.13)$$

in which  $[T]$  is a stress transformation matrix (Table 2.3). It can be shown that

$$[D] = [T] [D'] [T]^T \quad (2.14)$$

The table of direction cosines is given below:

Axes	1	2	3
R	$t_{11}$	$t_{12}$	$t_{13}$
$\theta$	$t_{21}$	$t_{22}$	$t_{23}$
z	$t_{31}$	$t_{32}$	$t_{33}$

where

$$\begin{aligned}
 t_{11} &= \cos \alpha_2 \sin(\alpha_1 - \theta_s \theta') \\
 t_{21} &= -\cos \alpha_2 \cos(\alpha_1 - \theta_s \theta') \\
 t_{31} &= -\sin \alpha_2 \\
 t_{12} &= \cos \alpha_3 \cos(\alpha_1 - \theta_s \theta') + \sin \alpha_3 \sin \alpha_2 \sin(\alpha_1 - \theta_s \theta') \\
 t_{22} &= \cos \alpha_3 \sin(\alpha_1 - \theta_s \theta') - \sin \alpha_3 \sin \alpha_2 \cos(\alpha_1 - \theta_s \theta') \\
 t_{32} &= \sin \alpha_3 \cos \alpha_2 \\
 t_{13} &= \cos \alpha_3 \sin \alpha_2 \sin(\alpha_1 - \theta_s \theta') - \sin \alpha_3 \cos(\alpha_1 - \theta_s \theta') \\
 t_{23} &= -\cos \alpha_3 \sin \alpha_2 \cos(\alpha_1 - \theta_s \theta') - \sin \alpha_3 \sin(\alpha_1 - \theta_s \theta') \\
 t_{33} &= \cos \alpha_3 \cos \alpha_2
 \end{aligned} \tag{2. 15}$$

The stiffness matrix of an element is

$$[k]^e = [c]^{-1T} \left( \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} [q]^T [D] [q] dV \right) [c]^{-1} \tag{2. 16}$$

in which

$$dV = r r_s \theta_s z_s dr' d\theta' dz'$$

The integral portion of eq. (2. 16) is evaluated by means of the Gaussian quadrature formula.

### 2. 3 Formulation of Essential Matrices for Two-Dimensional Elements

The same relationships as in the preceding section, but with terms involving  $z$  and  $w$  and components of stress and strain in the direction of Z-axis eliminated, are used as bases to derive the matrices for two-dimensional elements. The displacement expression becomes simply

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & r' & \theta' & r'\theta' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & r' & \theta' & r'\theta' \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{Bmatrix} \tag{2. 17}$$

The matrix  $[c]$  reduces to

$$\begin{bmatrix} t_{11}^2 & t_{12}^2 & t_{13}^2 & 2t_{11}t_{12} & 2t_{11}t_{13} & 2t_{12}t_{13} \\ t_{21}^2 & t_{22}^2 & t_{23}^2 & 2t_{21}t_{22} & 2t_{21}t_{23} & 2t_{22}t_{23} \\ t_{31}^2 & t_{32}^2 & t_{33}^2 & 2t_{31}t_{32} & 2t_{31}t_{33} & 2t_{32}t_{33} \\ t_{11}t_{21} & t_{12}t_{22} & t_{13}t_{23} & t_{12}t_{21} + t_{11}t_{22} & t_{13}t_{21} + t_{11}t_{23} & t_{12}t_{22} + t_{12}t_{23} \\ t_{11}t_{31} & t_{12}t_{32} & t_{13}t_{33} & t_{12}t_{31} + t_{11}t_{32} & t_{13}t_{31} + t_{11}t_{33} & t_{13}t_{32} + t_{12}t_{33} \\ t_{21}t_{31} & t_{22}t_{32} & t_{23}t_{33} & t_{22}t_{31} + t_{21}t_{32} & t_{23}t_{31} + t_{21}t_{33} & t_{23}t_{32} + t_{22}t_{33} \end{bmatrix}$$

Table 2.3 The Transformation Matrix [T]

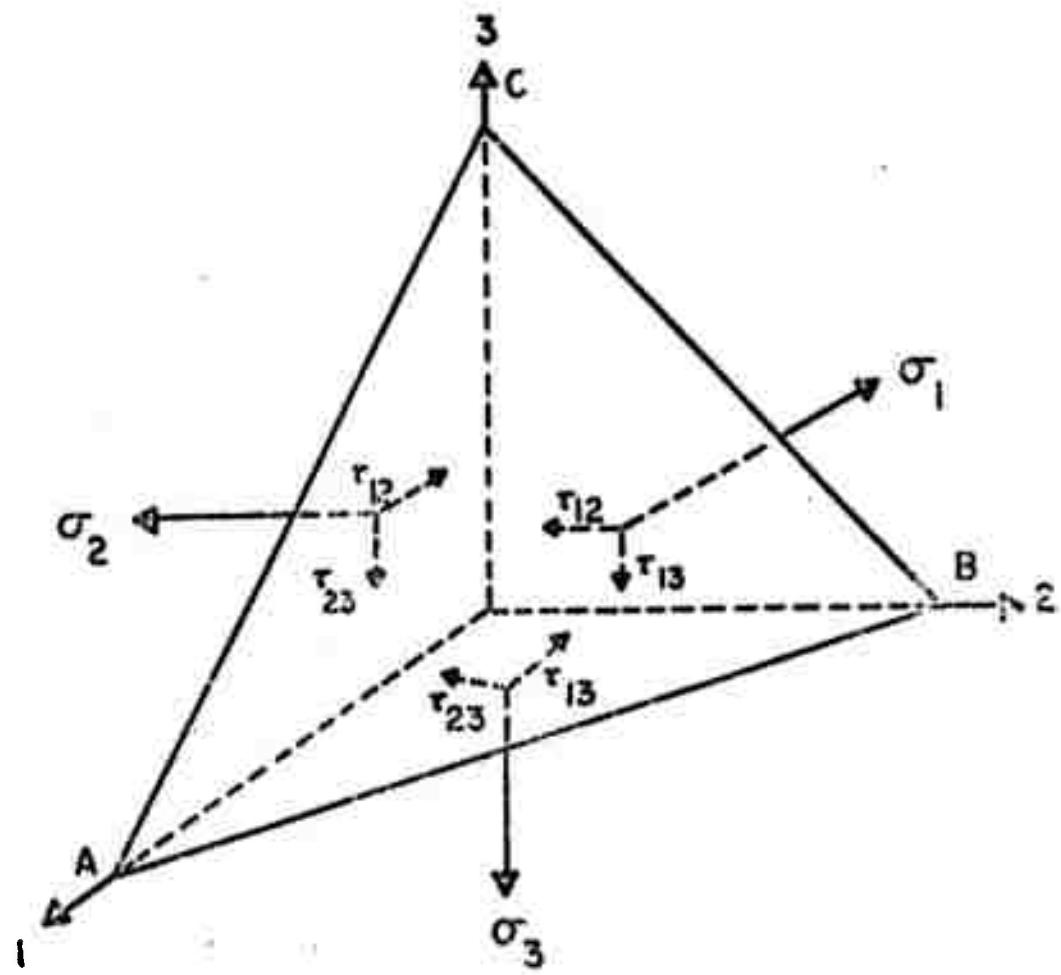
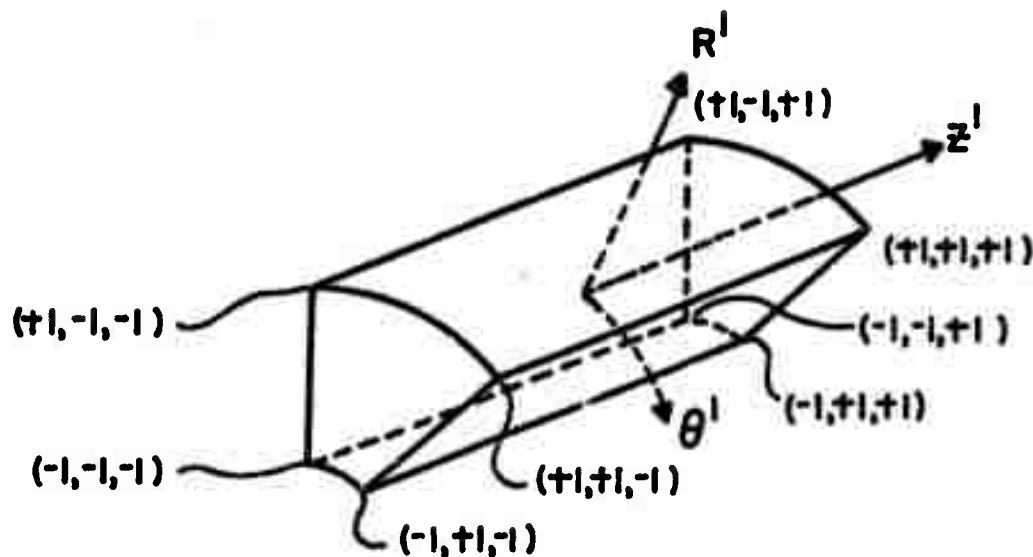
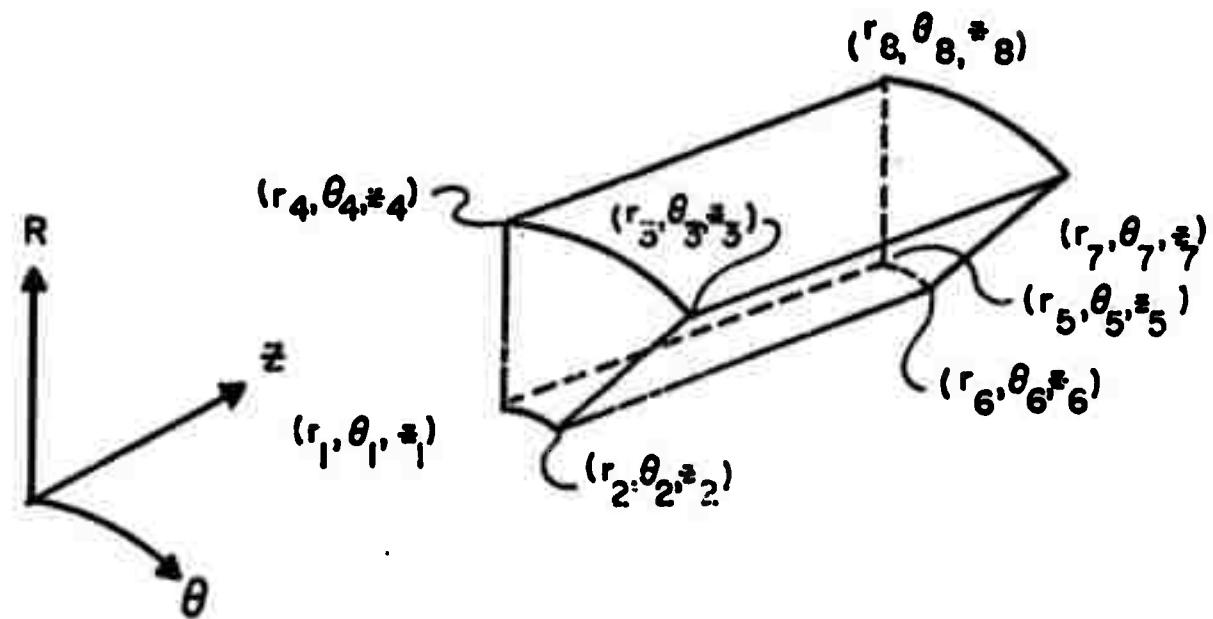


FIGURE 2.6 Stress Tetrahedron



(a) Local coordinates



(b) Global coordinates

FIGURE 2.7 Nodal Coordinates of Three-Dimensional Element

$$[c] = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix} \quad (2.18)$$

and its inverse becomes

$$[c]^{-1} = \frac{1}{4} [c]^T \quad (2.19)$$

The formulas for components of strain are now

$$(\epsilon) = \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{u}{r} + \frac{\partial v}{r \partial \theta} \\ \frac{\partial u}{r \partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \end{Bmatrix} \quad (2.20)$$

from which the matrix  $[q]$  is obtained to be

$$[q] = \begin{bmatrix} 0 & \frac{1}{r_s} & 0 & \frac{\theta}{r_s} & 0 & 0 & 0 & 0 \\ \frac{1}{r} & \frac{r'}{r} & \frac{\theta'}{r} & \frac{r'\theta'}{r} & 0 & 0 & \frac{1}{r\theta_s} & \frac{r'}{r\theta_s} \\ 0 & 0 & \frac{1}{r\theta_s} & \frac{r'}{r\theta_s} & -\frac{1}{r} & \frac{1}{r_s} - \frac{r'}{r} & -\frac{\theta'}{r} & \frac{\theta'}{r_s} - \frac{r\theta'}{r} \end{bmatrix} \quad (2.21)$$

The elasticity matrix for plane stress is

$$[D] = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \quad (2.22)$$

Similarly, for anisotropic case

$$[\mathbf{D}']^{-1} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu}{E_2} & 0 \\ 0 & \frac{1}{E_2} & 0 \\ \text{Sym.} & & \frac{1}{G_{12}} \end{bmatrix} \quad (2.23)$$

With  $\{\sigma\}$  equal to  $\{\sigma_r \sigma_\theta \tau_{r\theta}\}$ ,  $\{\sigma'\}$  equal to  $\{\sigma_1 \sigma_2 \tau_{12}\}$  and  $\alpha_1$  defined as in section 2.2, the stress transformation matrix is calculated to be

$$[\mathbf{T}] = \begin{bmatrix} \sin^2 \alpha & \cos^2 \alpha & 2 \sin \alpha \cos \alpha \\ \cos^2 \alpha & \sin^2 \alpha & -2 \sin \alpha \cos \alpha \\ -\sin \alpha \cos \alpha & \sin \alpha \cos \alpha & \sin^2 \alpha - \cos^2 \alpha \end{bmatrix} \quad (2.24)$$

in which  $\alpha = \alpha_1 - \theta_s$ .

In the element stiffness expression [eq. (2.22)],

$$dV = r t r_s \theta_s dr d\theta \quad (2.25)$$

in which  $t$  is the thickness of the disc.

It should be noted that all the matrices, except  $[\mathbf{D}]$ , formed in this section could also have been obtained directly from the three-dimensional matrices of the preceding section by simply deleting appropriate rows and columns of the latter and setting values of certain parameters to zero. In particular, to obtain  $[\mathbf{c}]$ , rows 3, 6, 9 and 12 to 24 and columns 5 to 8 and 13 to 24 of Table 2.1 are deleted; to obtain  $[\mathbf{q}]$  rows 3, 5 and 6 and columns 5 to 8 and 13 to 24 of Table 2.2 are deleted; to obtain  $[\mathbf{T}]$  rows and columns 3, 5 and 6 of Table 2.3 are deleted and  $\alpha_2$  and  $\alpha_3$  are set to zero. Thus, by adding a few IF statements to and generalizing some indices of a three-dimensional computer program, the program can be made to work for two-dimensional cases as well.

## 2.4 Failure Criteria

The analysis described in this chapter will be based on any one or combination of the following failure criteria:

### 1. Maximum principal stress theory--

Under this criterion, an element will be considered failed if one of the principal stresses equals or exceeds the strength (elastic limit or yield point) of the material making up the element. If tensile failure occurs, the modulus of elasticity across the crack is reduced to zero and the stiffness of the element is revised accordingly. The element can therefore no longer resist tensile stress (no stress reversal is anticipated) normal to the crack but can still take tension or compression in the other directions.

If failure occurs in compression (crushing), an approximation is made. To account for the fact that rock exhibits ductility when subjected to high confining pressure, only a portion of the stiffness of the failed element will be removed. A more realistic approach would be to undertake inelastic analysis to obtain the actual stiffness of the failed element. This approach, however, presupposes the availability of a stress-strain curve which might not be feasible in most rocks. Indeed, such curve can only be obtained through a triaxial test and its shape will vary widely depending on, among other factors, the rock type, shape of specimen, and intensity of confining pressure. Stress-strain curves of few rocks are available (1) but only at certain confining pressure intensities.

### 2. Maximum shearing stress theory--

This criterion states that failure occurs when the maximum shearing stress in a material equals or exceeds the critical shearing stress. The Coulomb-Navier (1) version will be used. This states that shearing failure will occur when

$$\sigma_1(-\mu + \sqrt{\mu^2 + 1}) - \sigma_3(\mu + \sqrt{\mu^2 + 1}) \geq 2\tau_{cr} \quad (2.26)$$

in which  $\tau_{cr}$  is the critical shearing stress found to be between 2% and 15% of uniaxial compressive strength;  $\mu$  is the coefficient of internal friction; and  $\sigma_1$  and  $\sigma_3$  are the major and minor principal stresses. The above expression takes into account frictional resistance to sliding along the failure plane.

It should be noted that if  $\mu = 0$ , that is, there is no frictional resistance, the above inequality reduces to

$$\sigma_1 - \sigma_3 \geq 2 \tau_{cr} \quad (2.27)$$

in which the left side is simply the expression for twice the maximum shearing stress at a point.

Again, post-failure behavior of elements will be approximated.

### 3. Maximum principal strain theory--

This could be a better alternative to the first criterion in the sense that here effects of the other principal stresses acting normal to the direction being investigated are considered. To illustrate this point, consider Fig. 2.8. Let  $\sigma_e$  be the critical stress and  $\epsilon_e$  the corresponding strain. In the first figure, both the first and the present criteria will predict the same failure stress, namely,  $\sigma_1 = \sigma_e$ . In the second figure, the first criterion will predict the failure condition  $\sigma_1 = \sigma_e$ , but the present criterion will predict a value greater than  $\sigma_e$  since failure occurs when  $(\sigma_1 - v\sigma_2) \geq \epsilon_e E$ . Similarly, a value less than  $\sigma_e$  will be predicted if  $\sigma_2$  is in tension.

Failed elements will be treated the same way as in the first criterion.

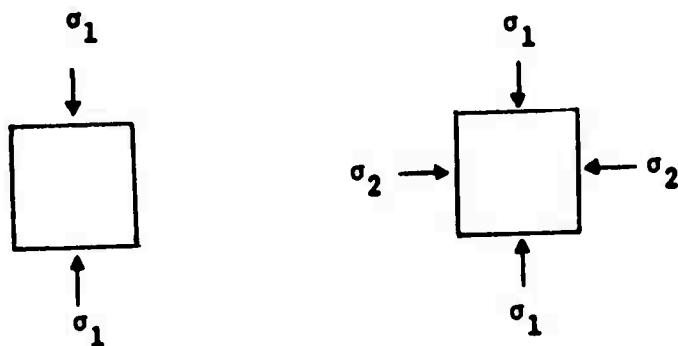


Figure 2.8

## 2.5 Concluding Remarks

Derived in this chapter are the different matrices needed in the development of the finite element program proposed in this report. Discussion of such other integral parts of the program as (a) building up of the global stiffness matrix, (b) determination of nodal displacements and stresses, and (c) calculation of principal stresses is omitted because they are discussed in detail elsewhere (2, 3, 5, 6).

An equation solver proposed by Jensen and Parks (6) will be used in the program. The solver contains an optimal nodal renumbering scheme to conserve sparseness of the stiffness matrix. Only nonzero terms of the matrix are stored and processed.

## REFERENCES

1. Farmer, I. W., Engineering Properties of Rocks, E & FN Spon Ltd., London, 1968.
2. Timoshenko, S. and J. N. Goodier, Theory of Elasticity, McGraw-Hill Book Company, Inc., New York, 1951.
3. Zienkiewicz, O. C. and Y. K. Cheung, The Finite Element Method in Structural and Continuum Mechanics, McGraw-Hill Book Co., London, 1967.
4. Zienkiewicz, O. C., S. Valliappan, and I. P. King, "Stress Analysis of Rock as a 'No Tension' Material," Geotechnique, Vol. 18, March 1968, pp. 56-66.
5. Basas, J. E. "Static Analysis of Stiffened Shells by the Finite Element Method," Ph. D. Dissertation, Department of Civil Engineering, University of Wisconsin, Madison, Wisconsin, 1971.
6. Jensen, H. G. and G. A. Parks, "Efficient Solutions for Linear Matrix Equations," Journal of the Structural Division, Proc. ASCE, Vol. 96, No. ST1, Jan. 1970, pp. 49-64.
7. Seely, F. B. and J. O. Smith, Advanced Mechanics of Materials, John Wiley & Sons, Inc., New York, 1932.

## CHAPTER 3

### TWO-DIMENSIONAL PROGRAM

#### 3. 1 Introduction

The program described herein was developed as an alternative to the two-dimensional program submitted as part of the first annual progress report (1). It contains the following features not found in the earlier program:

1. Element and node information are generated automatically thus avoiding the preparation of voluminous deck of input cards.
2. Element sizes may vary, allowing for finer mesh in regions where stresses are large and coarser mesh elsewhere, for a more efficient solution.
3. More than one element may fail during each loading cycle.

No change is made in the failure criteria.

The element used in the present program is shown in Fig. 2. 1 and described in section 2. 3. The element belongs to the so-called "isoparametric" group.

#### 3. 2 Description of Program

The flow of the program is illustrated in Fig. 3. 2. The total program is made up of the main program and four subroutines (complete listing is shown in Appendix A). Each performs the following tasks:

##### Main Program

- generates element and node numbers in the sequence shown in Fig. 3. 1;
- generates the nodal coordinates, given the radial coordinates of the rings and the circumferential coordinates of the radial lines;
- forms the global stiffness matrix in blocks;
- calculates the stresses and load factors (ratio of allowable stress to principal stress) of each element;

-- calculates critical loads, displacements and strains; and  
-- removes 90% of stiffness matrix of failed elements from global  
stiffness matrix.

#### Subroutine DISPL

-- evaluates the unknown nodal displacements by the Gaussian  
elimination method and back-substitution.

#### Subroutine MATB

-- forms the matrix [q] and the matrix product [D][q].

#### Subroutine INTEG

-- forms the stiffness matrix of the elements, employing Gaussian  
quadrature formulas in place of actual analytical integration.

#### Subroutine UNIFRM

-- generates elastic properties of elements of nonhomogeneous disc by  
means of the random number routine RANUN (2).

RANUN generates arbitrary random numbers assuming statistically  
uniform distribution. The first generative number of RANUN can be set to  
any number N by a call to RANUNS(N). Both RANUN and RANUNS are Madison  
Academic Computing Center (MACC) library routines.

It should be noted that in numbering the nodes, the center point of  
the disc is multinumbered (see Fig. 3. 1). This is done because each element  
has to have four nodes.

The well-known banded matrix technique in solving simultaneous  
linear equations is employed in the program.

### 3. 3 Test Problems

Several problems were run to test the correctness of the program. The  
results of three are presented here.

#### A. Test Problem No. 1

A disc, 20 inches in diameter and 1 inch thick, is analyzed. It is  
required to determine the stress distribution along the line of the load.

The disc is assumed homogeneous and isotropic.

Because of symmetry, only a quarter of the disc is considered in the analysis. The quarter disc is divided into 25 rings and 18 equal slices for a total of 408 elements and 450 nodes. The results are shown in Fig. 3.3.

#### B. Test Problem No. 2 (Homogeneous Case)

A homogeneous disc, 3 inches in diameter, 1 inch thick, and possessing the following elastic properties

Modulus of elasticity :  $5.7 \times 10^6$  psi

Poisson's ratio : 0.25

Allowable compression: 27,000 psi

Allowable tension : 5% of allowable compression

is analyzed. The disc is divided into the mesh shown in Fig. 3.4. A listing of the required input data is printed on page 50. Some of the results of the analysis are indicated in the following pages.

#### C. Test Problem No.3 (Nonhomogeneous Case)

A nonhomogeneous disc, 3 inches in diameter and 1 inch thick, is analyzed. The elastic properties of the elements vary as follows:

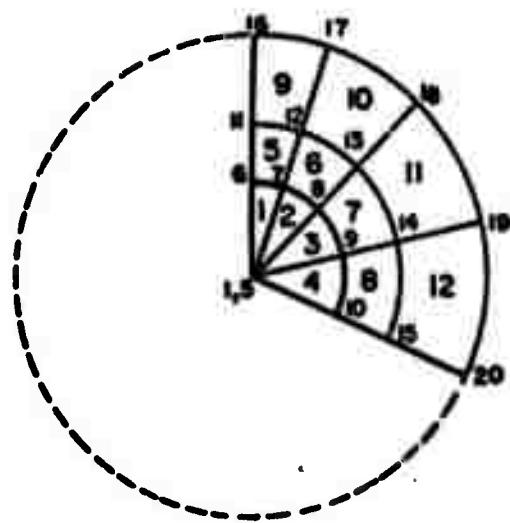
Modulus of elasticity :  $5.5 \times 10^6$  to  $6.5 \times 10^6$  psi

Poisson's ratio : 0.23 to 0.27

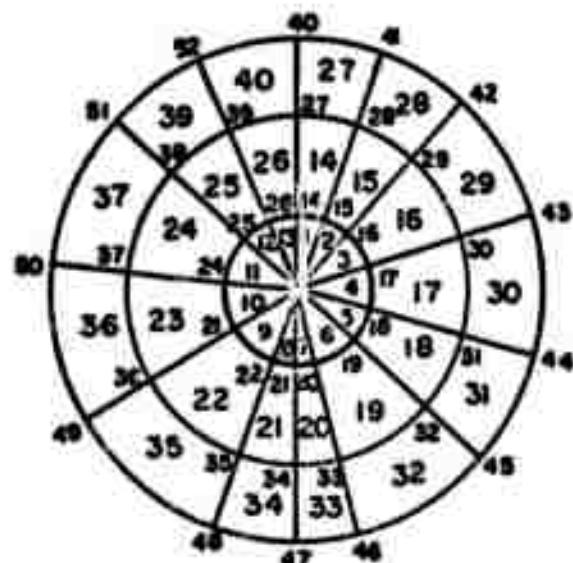
Allowable compression: 22,000 to 32,000 psi

(allowable tension = 5% of allowable compression)

Double symmetry is assumed to avoid having to analyze the entire disc which would require considerably higher computer expense. The disc is divided into the same mesh shown in Fig. 3.4. Several runs were made. During each run, a different starting point for the random number generator was specified. The necessary input is listed on page 51. Results of the analysis are indicated in the following pages.

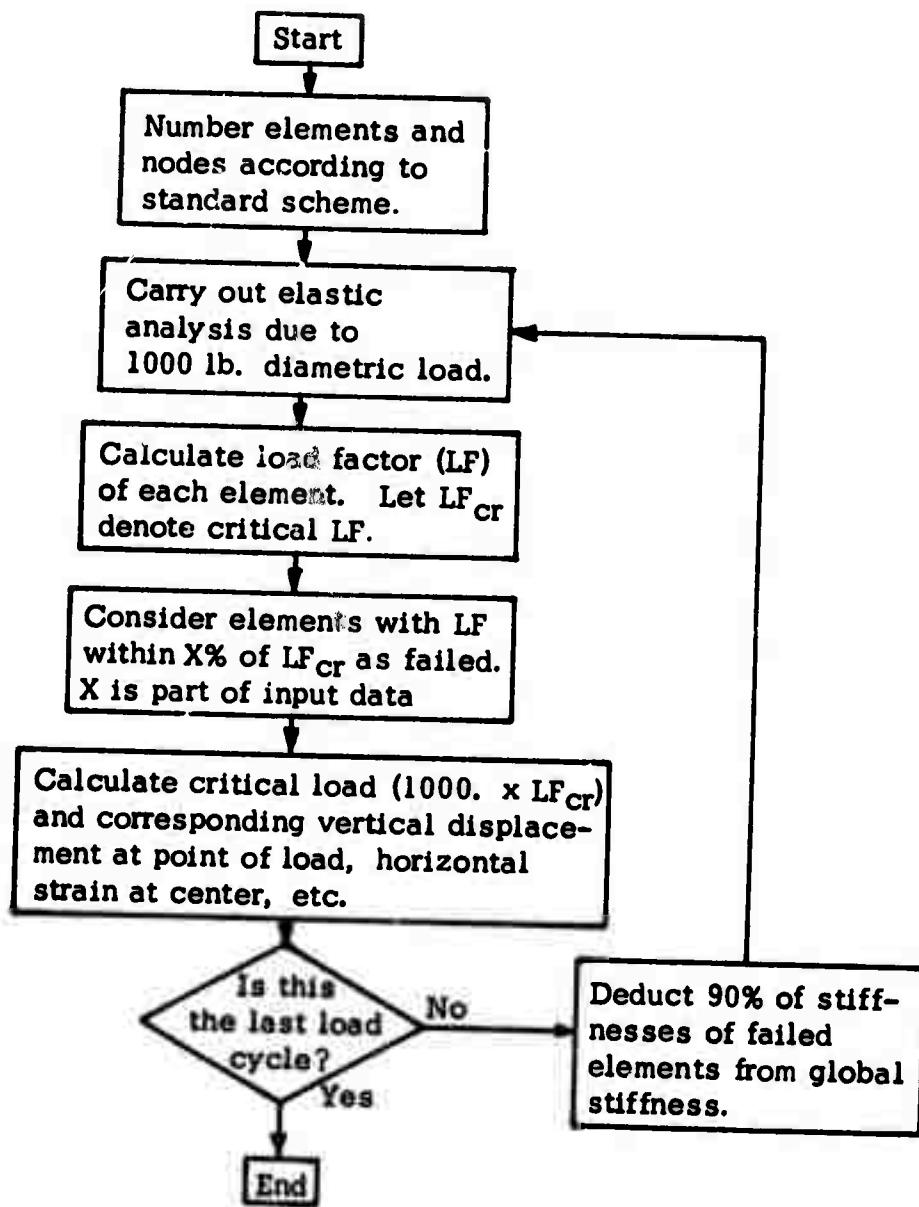


(a) Partial Disc



(b) Full Disc

**FIGURE 3.1 Standard Scheme for Numbering Nodes and Elements of Two-Dimensional Disc**



Note: Load factor is the ratio of allowable stress to principal stress.

**FIGURE 3. 2 FLOW CHART OF TWO-DIMENSIONAL PROGRAM.**

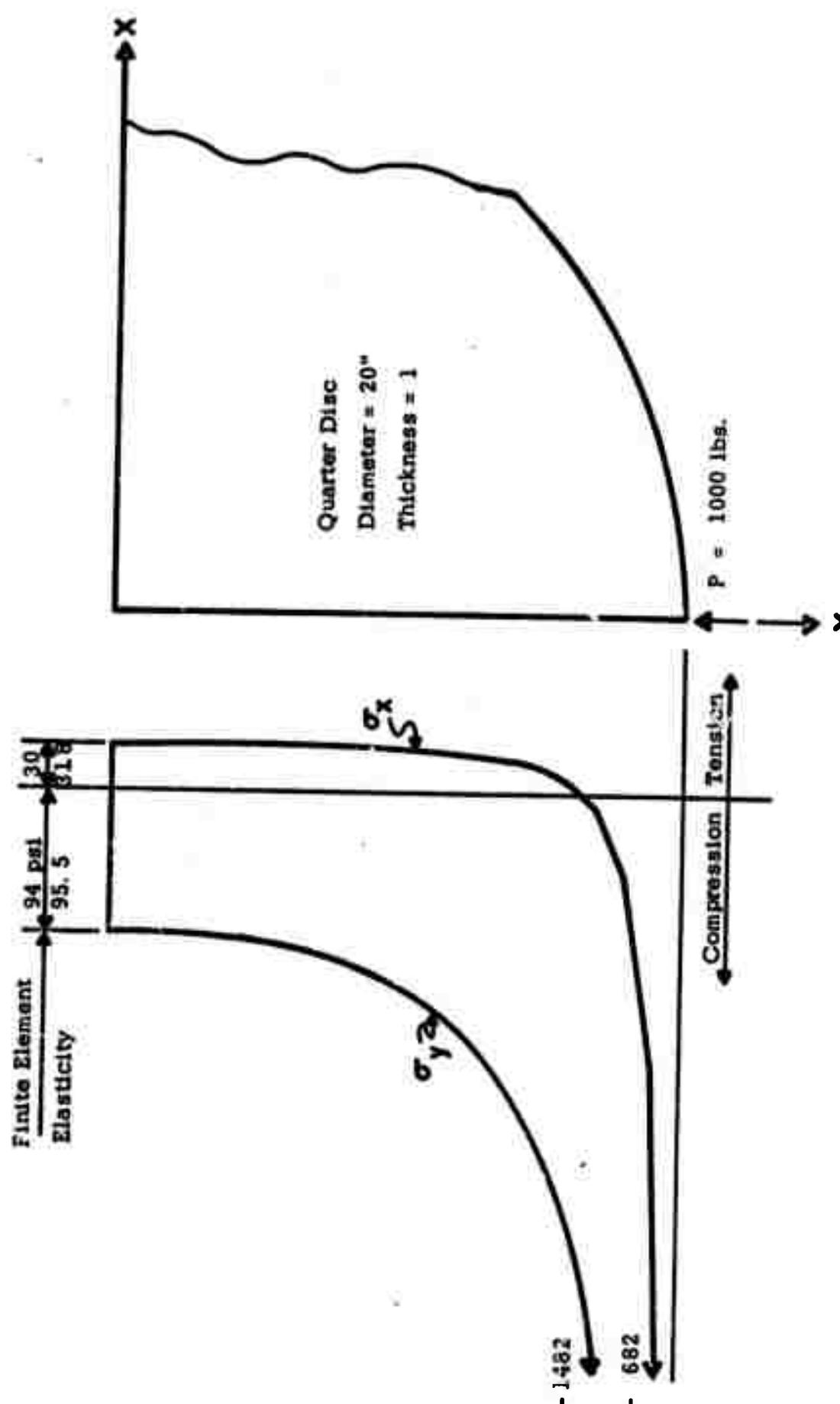
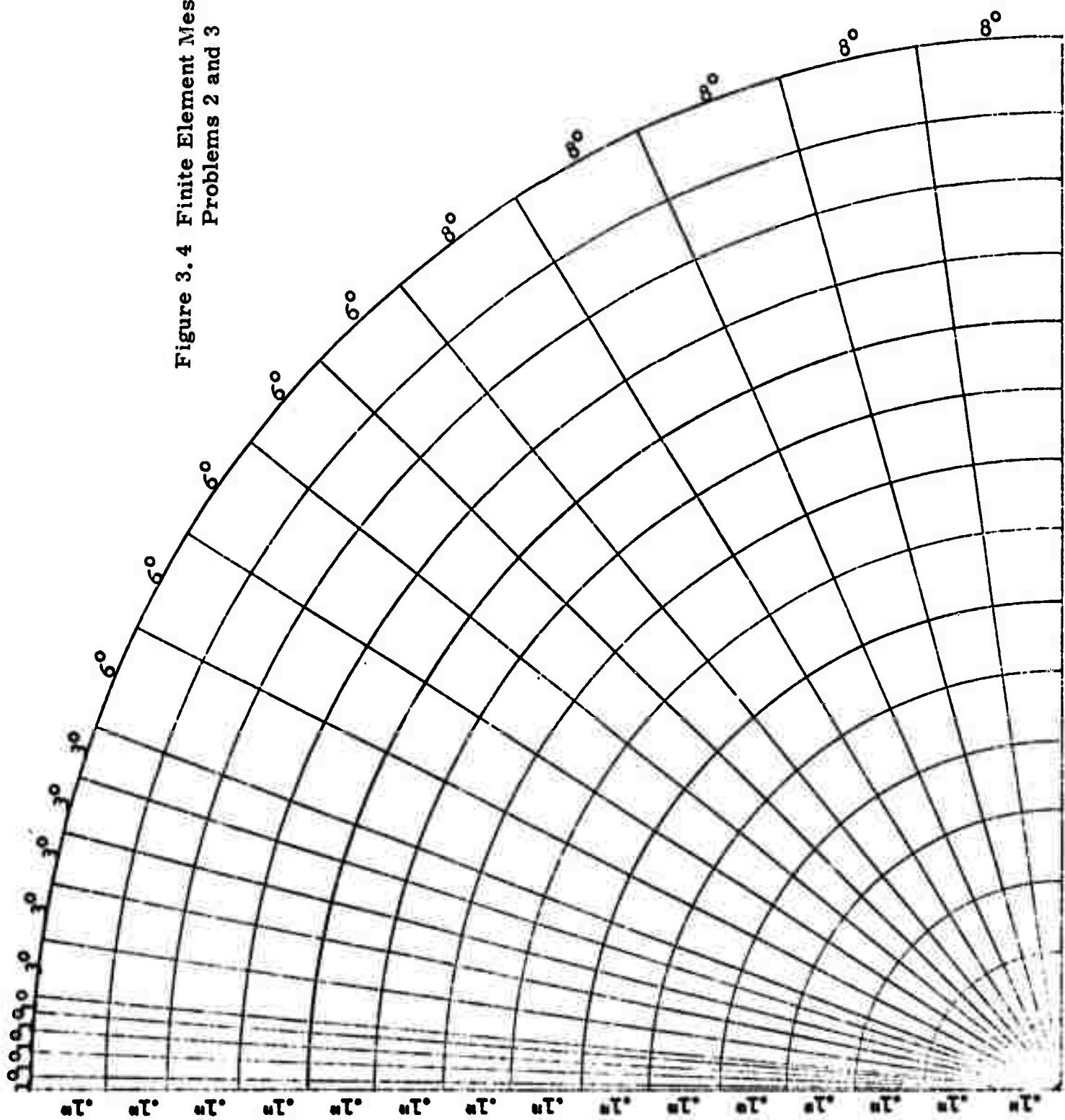


FIGURE 3.3 Stress Distribution Along Line of Diametral Load (Test Problem No. 1)

**Figure 3.4** Finite Element Mesh for Problems 2 and 3



**INPUT DATA FOR TEST PROBLEM NO. 2 (HOMOGENEOUS CASE)**

Column	Card No.
1	5 6 10 11 15 16 20 21 25 26 30 31 35 36 40 41 45 46 50 51 55 56 60 61 65 66
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	

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INPUT DATA FOR TEST PROBLEM NO. 3 (NONHOMOGENEOUS CASE)

### 3. 4 Discussion of Results

The results from Test Problem No. 1 clearly demonstrate the theoretical correctness of the finite element solution formulated in this report. As can be seen from Fig. 3. 3, the elasticity and finite element solutions differ by only 6% for  $\sigma_x$  and 2% for  $\sigma_y$  at the center of the disc. By solving the same problem several times progressively decreasing element size each time, the finite element solution was also found to be convergent.

The discs analyzed in Test Problems No. 2 and No. 3 are models for the Valders limestones studied extensively in the experimental phase of this research (Chapter 1). The elastic properties assigned to the nonhomogeneous disc are actual range of values obtained from experiment, while those assigned to the homogeneous disc are the mean of these values. The failure pattern predicted by the finite element solution (Figs. 3. 5 and 3. 6) agrees favorably with the actual failure patterns of Brazilian tests. The critical load, however, appears to be underestimated--5730 lbs. and 4400 lbs. for homogeneous and nonhomogeneous cases compared to the 6940 lbs. average obtained from the experiment. This seems to indicate that either the 90% factor used to deduct the stiffness matrix of failed elements is too high or the failure mechanism assumed in the program is not altogether realistic or both.

The load-displacement curves appear to follow the same general shape regardless of material characteristics or starting point of random number generator (see Figs. 3. 7 and 3. 8). These curves compare favorable with those obtained from the experiment at the early stages of loading.

### 3. 5 Conclusions

Based on the several computer runs made, the following conclusions are drawn:

1. Even in its oversimplified form, the finite element solution is capable of predicting the actual failure pattern of Brazilian tests.
2. With a few improvements in the failure criteria, there is a strong

possibility that the critical load and the load-displacement curve can be accurately predicted as well.

3. In nonhomogeneous cases, the shape of the load-displacement curve is not affected significantly by the choice of the starting point of the random number generator.

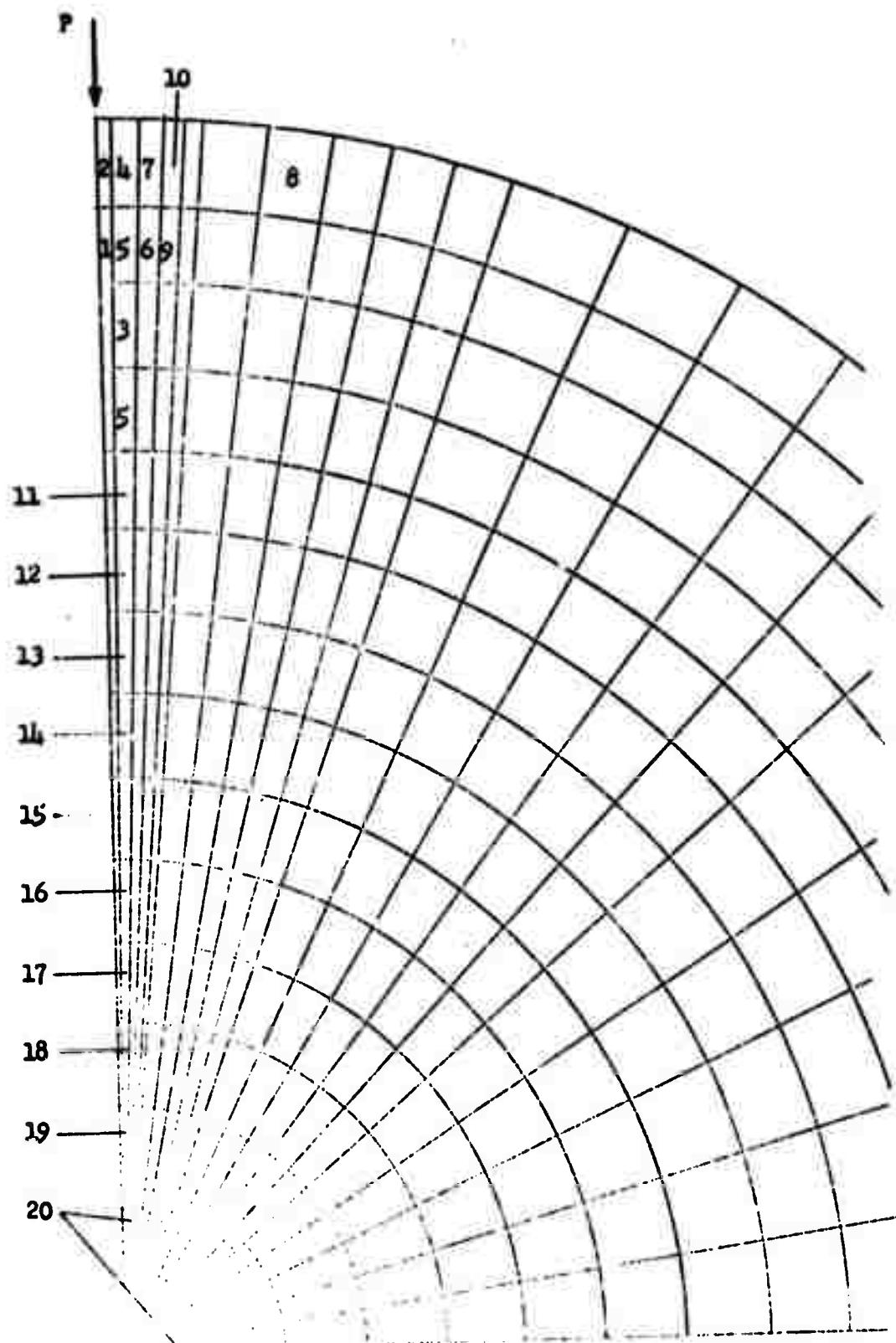


Figure 3.5 Progression of Failure in Test Problem No. 2

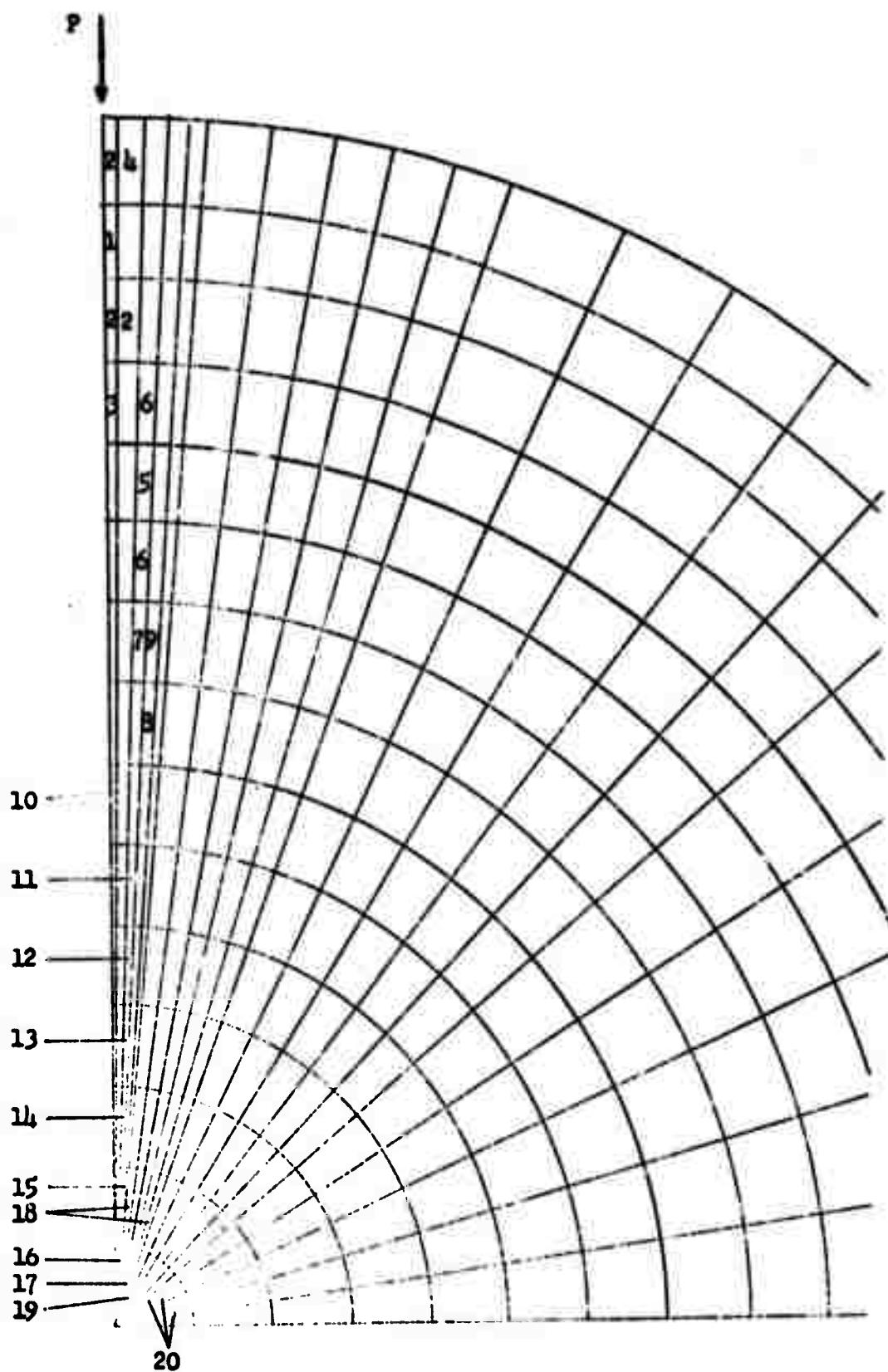


Figure 3.6 Progression of Failure in Test Problem No. 3

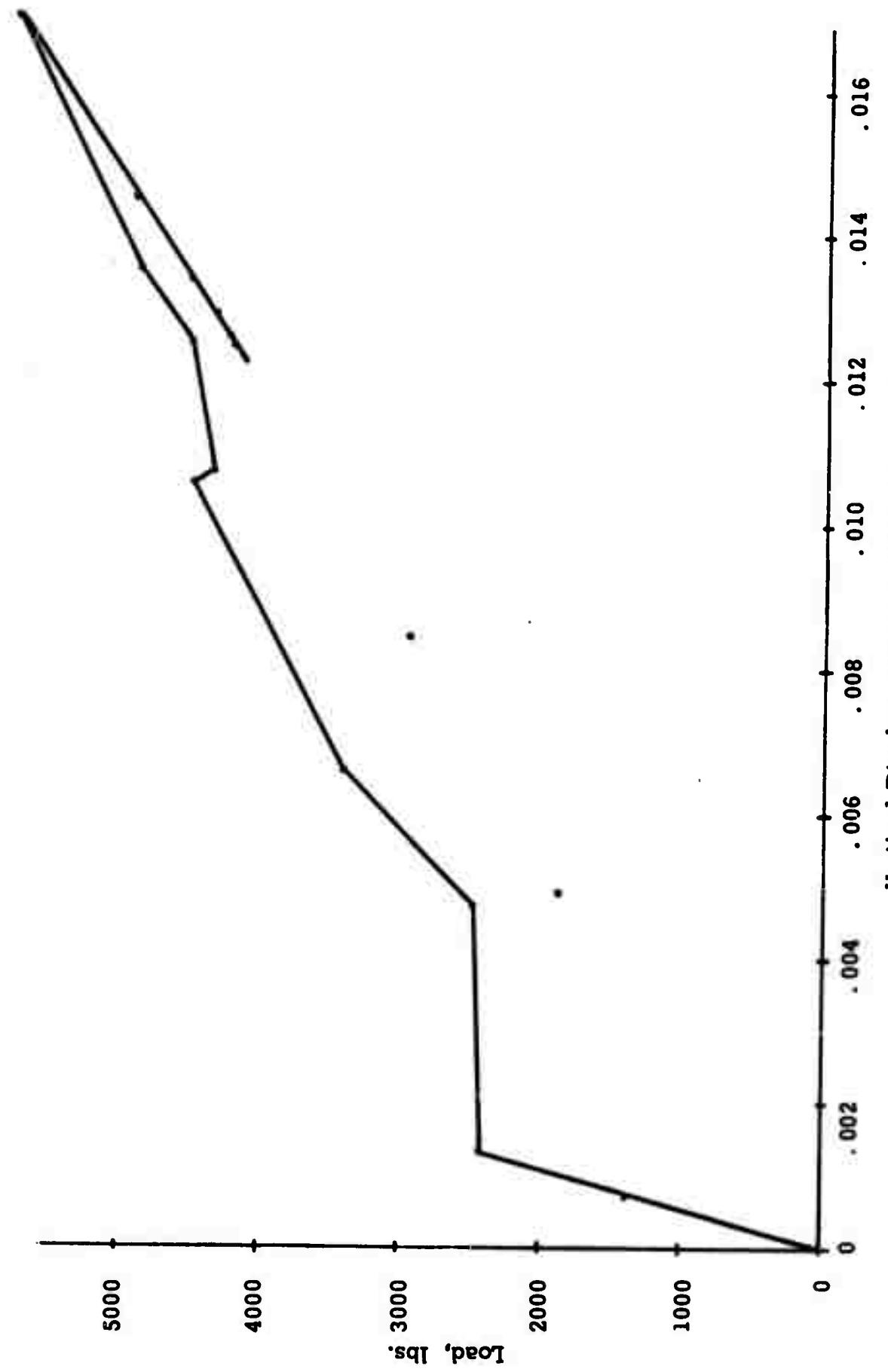


FIGURE 3.7 Load-Displacement Curve (Homogeneous Case)

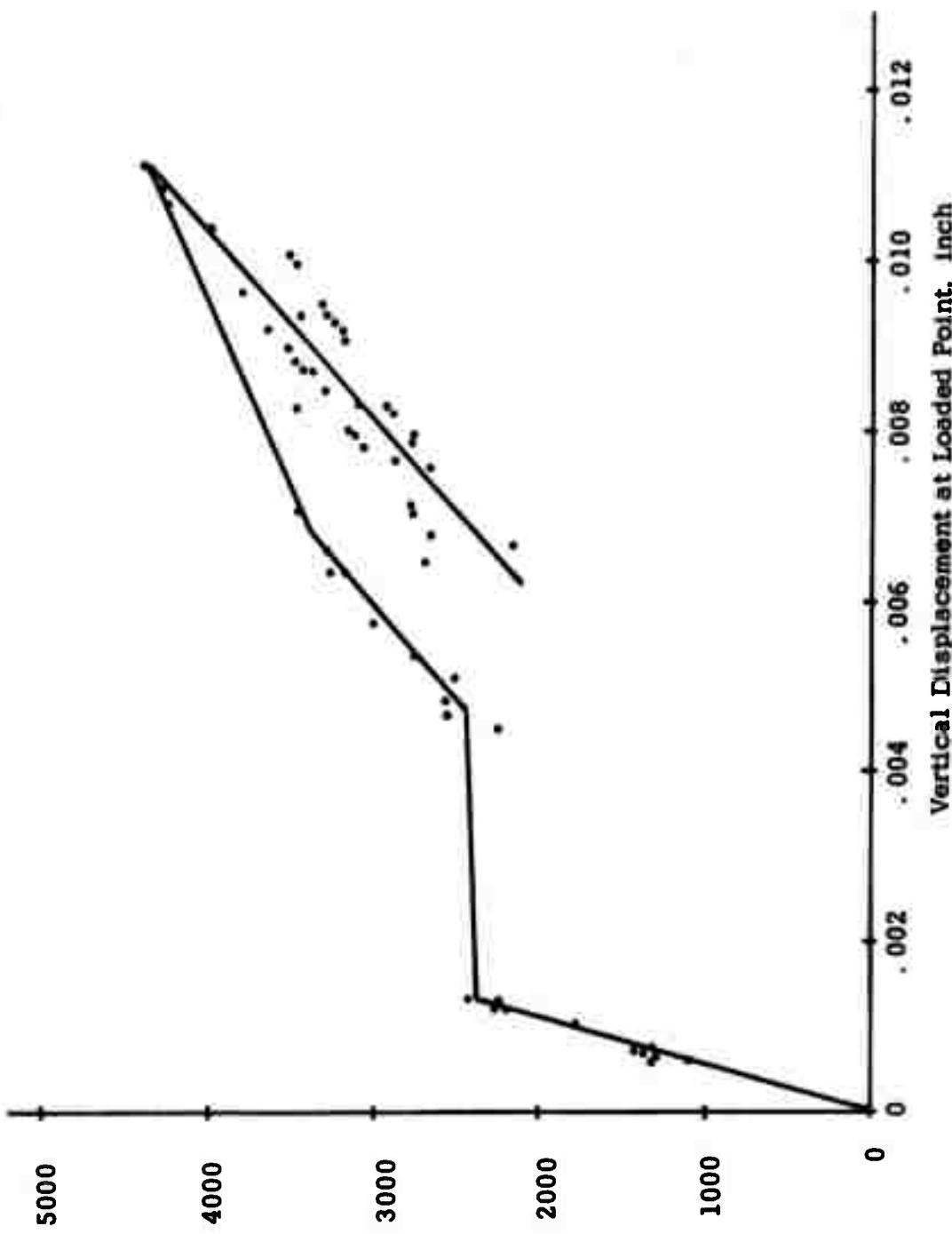


FIGURE 3.8 Load-Displacement Curve (Nonhomogeneous Case)

## REFERENCES

1. Aspects of Mechanical Behavior of Rock Under Static and Cyclic Loading (Part A--Mechanical Behavior of Rock under Static Loading), Annual Technical Progress Report, Engineering Experiment Station, University of Wisconsin - Madison, March 1972.
2. Madison Academic Computing Center, Random Number Routines, Mathematical Routines Series, October 1969.
3. Basas, J. E., Static Analysis of Stiffened Shells by the Finite Element Method, Ph. D. Dissertation, Department of Civil Engineering, University of Wisconsin, Madison, Wisconsin, 1971.

## APPENDIX A

### TWO-DIMENSIONAL PROGRAM LISTING

The computer program presented here is written in Fortran V language specifically for the Univac 1108 computer at the University of Wisconsin - Madison. The program is capable of handling all sorts of static two-dimensional loadings, not just the diametric loading of the Brazilian test. Zero or nonzero displacements may be assigned to any node making possible solutions involving portions of the disc only.

The program requires usage of three auxiliary storage tapes or drums designated by the numbers 10, 11 and 12.

The listing is complete except for the random number routines RANUN(R) and RANUNS(N). These subprograms are provided by the Madison Academic Computing Center (MACC).

A description of the input and output parameters required in the program is given below.

#### Input Notations:

- |                 |  |
|-----------------|--|
| 1. (a) NCHO     | - Printout code; 0 if only stresses at failed elements are to be printed out; nonzero if stresses at all elements are to be printed out.         |
| (b) NWCL        | - Node at which radial displacement is to be computed.   |
| (c) NST1, NST2  | - Nodes between which strain is to be computed.  |
| 2. (a) EM1, EM2 | - Range of values of elastic moduli.   |
| (b) CS1, CS2    | - Range of values of allowable compressive stresses.   |
| (c) PR1, PR2    | - Range of values of Poisson's ratios.   |
| (d) CTRAT       | - Ratio of allowable tension to allowable compression.   |
| (e) T           | - Thickness of disc.   |
| (f) TRN         | - Maximum percentage variation from largest load factor for element to be considered failed; 0 if only one element is allowed to fail each time. |

- (g) DIA                    - Diameter of disc.
- 3. (a) IRN                - Total number of elements in vicinity of applied loads.
- (b) NIK(I)                - Number of Ith element in vicinity of applied loads; specified if run is to be terminated upon failure of element; 0 if run is to continue.
- 4. (a) NELEM              - Total number of elements.
- (b) NTNN                 - Total number of nodes.
- (c) NNSD                - Total number of nodes with prescribed displacements.
- (d) NNCL                - Total number of nodes at which concentrated loads are applied.
- (e) NE                    - Total number of slices portion of disc involved in analysis is divided into.
- (f) NS                    - Total number of equal slices whole disc is divided into; 0 if disc is not divided into equal slices and only a portion of whole disc is involved in analysis; any negative number if disc is not divided into equal slices and whole disc is involved in analysis.
- (g) NC                    - Total number of rings plus one disc is divided into.
- (h) NER                 - Least total number of similar consecutively numbered elements; 1 in nonhomogeneous and anisotropic problems. This parameter allows the generation of stiffness matrix of similar elements only once.
- (i) NCOD                - Total number of load cycles.
- 5. N                        - First generative number of random number routine. (Note that this information is supplied only in nonhomogeneous problems.)
- 6. RADN(I)                - Radial coordinates of rings starting with 0.
- 7. TTAN(I)                - Circumferential coordinates in degrees of radial lines dividing disc into slices. (Note that this information is supplied only when disc is divided into unequal slices.)

8. (a) MR                    - 1 for 1st card; 2 for 2nd card; 3 for 3rd card, etc.
- (b) NODC(MR)            - Number of MRth node with externally applied concentrated load.
- (c) CLOAD(MR, I)        - Magnitude of concentrated load. (I = 1 in radial direction; I = 2 in circumferential direction.)
9. (a) NODB(IX)            - Number of IXth node with prescribed displacement.
- (b) DISP(IX, I)          - Value of prescribed displacement; 200. if displacement is not prescribed. (I = 1 in radial direction; I = 2 in circumferential direction.)
- (c) JJ                    - Total number of nodes which exactly have the same prescribed displacements as NODB and which form an arithmetic progression with NODB; 0 if none such nodes.
- (d) II                    - Interval of arithmetic progression.

Output Notations:

- RADIAL                    - Direct stress in radial direction.
- CIRCUM                    - Direct stress in circumferential direction.
- SHEAR                    - Shearing stress.
- PRNCP1, PRNCP2        - Principal stresses.
- MAXSHR                    - Maximum shearing stress.
- LD FAC                    - Load factor (ratio of allowable stress to actual stress).

All other output notations are self-explanatory.

```

1.      C      ANALYSIS OF CIRCULAR DISC
2.      COMMON/JACKP/C14,N1,P1,T,2),FL(9),SL(8,8),RR(3,8)
3.      COMMON/COLOC/NTNN,NLW,NEN,NL2,F(200),S(200,100)
4.      COMMON/BOPAF/FP,F,T,TCR,CCR
5.      COMMON/TNATK/TTA(FU),RAD(FU),NOD(FU,4)
6.      COMMON/KRING/X1,X2,Y1,Y2,X(3),Y(3),Z(3),MN
7.      DIMENSION NODC(100),CLOAD(100,2),NODF(150),
8.      IDTSF(150,2),RADN(100),TTAN(100),GA(F),CR(E),
9.      COL(E),FD(E),SI,ZZ(55),F1,FLDF(55),
10.     TNFL(FU),KEF(55),N7K(20)
11.     DATA (CA(I),I=1,5)/-1.0,-1.0,1.0,0.0/
12.     DATA (GA(I),I=1,5)/-1.0,1.0,1.0,-1.0,0.0/
13.     DATA (CI1,T1),T=1,4)/-0.25,-0.25,-0.25,-0.25/
14.     DATA (CI2,T1),T=1,4)/-0.25,-0.25,-0.25,-0.25/
15.     DATA (CI3,T1),T=1,4)/-0.25,-0.25,-0.25,-0.25/
16.     DATA (CI4,T1),T=1,4)/-0.25,-0.25,-0.25,-0.25/
17.     DD 26 I=1,3
18.     DD 30 J=1,8
19.     26 R(I,J)=0.
20.     READ 1005,NCHO,NWCL,NST1,NST2
21.     1005 FORMAT (4I5)
22.     READ 1006,FM1,FM2,CS1,CS2,PR1,PR2,CTRAT,T,TPN,DIA
23.     1006 FORMAT (4F10.3,FF5.1)
24.     PRINT 1006,FM1,FM2
25.     1007 FORMAT (1SH1ELASTIC MODULUS = ,1PE12.E,4H TO ,1FF12.F)
26.     PRINT 1007,PR1,PR2
27.     1007 FORMAT (1SH1EISSONS RATIO = ,F10.5+4H TO ,F10.5)
28.     PRINT 1008,CS1,CS2
29.     1008 FORMAT (1SH1EITICAL COMPRESSIVE STRESS = ,
30.                  1PF10.3+4H TO ,PF10.3)
31.     PRINT 1008,CTRAT
32.     1010 FORMAT (1SH1CITICAL TENSION/COMPRESSION RATIO = ,FF.2)
33.     PRINT 1010,1
34.     1011 FORMAT (1SH1THICKNESS OF DISC = ,FE.2)
35.     PRINT 1011,DIA
36.     1011 FORMAT (1SH1DIAMETER OF DISC = ,FS.2)
37.     READ 1001,INH,INTK(1),I=1,TRN1
38.     READ 1001,NFLN,NTNN,NSD,NNCL,NS,NC,NFR,NCDD
39.     1001 FORMAT (1I1)
40.     NC1=NC-1
41.     PRINT 1012,NFLN
42.     1012 FORMAT (1SH1TOTAL NUMBER OF ELEMENTS = ,I3)
43.     PRINT 1012,NTNN
44.     1013 FORMAT (1SH1TOTAL NUMBER OF NODES = ,I3)
45.     PRINT 1013,NC
46.     1014 FORMAT (1SH1DISC OR PORTION OF DISC IS DIVIDED INTO ,
47.                  1I6+1H SLICES AND ,1I6+1H RINGS)
48.     1014 FORMAT (1SH1(TOTAL NUMBER OF ELEMENTS = ,I3,I3,I3,I3))
49.     1014 READ 1014,N
50.     1152 FORMAT (1I1)
51.     CALL RADUN(1)
52.     DD 100 T=1,NFLN
53.     CALL UNTRM (FM1,FM2,FS1)
54.     CALL UNTRM (PP1,PR2,PR1)
55.     CALL UNTRM (CS1,CS2,CS1)

```



```

FF.      DD(7,2)E15
FF.      DD(7,2)E15
TF.      100  DD(7,3)E-CSS
FF.      PRINT 11E2,PD(2,1),PD(21,1)
FU.      11E9  FORMAT (1FF12.6,1FF12.6)
F1.      GO TO 117
F2.      137  DO 101 T=1,NFLFM
F3.      DD(7,1)E15
F4.      DD(7,1)E15
F5.      101  DD(7,3)E-CSS
FF.      102  PRINT E997
F7.      E997 FORMAT (3H0FLFMNT,6X,12HCORNER NODES,3X,8H ELEMENT,
FF.      11X,12HCORNER NODES)
FU.      7010  FORMAT (5T6)
TF.      1F (NF-NS19I+21+13F
71.      13F  TF (NS19I+99+3H
72.      9I  DO 200 T=1,NFLFM
73.      NOD(I,T,1)=I+(T-1)*NF
74.      NOD(I,T,2)=NOD(I,T,1)+1
75.      NOD(I,T,3)=NOD(I,T,2)+1*NE
76.      NOD(I,T,4)=NOD(I,T,3)-1
77.      200  CONTINUE
78.      NPHE=(NE+3)*3
79.      GO TO 150
80.      51  DO 199 I=1,NFLFM
81.      NOD(I,1)=I
82.      NOD(I,2)=I+1
83.      NOD(I,3)=NOD(I,T,2)+NE
84.      NOD(I,4)=NOD(I,T,3)-1
85.      199  CONTINUE
86.      NRHENE=4
87.      DO 198 I=NF+1,NFLFM+1,NE
88.      NOD(I,1)=NOD(I,T,2)-NF
89.      NOD(I,3)=NOD(I,T,3)-NF
90.      198  CONTINUE
91.      170  NG=NFLFM/2
92.      NC1=(NFLFM+1)/2
93.      DO 23 I=1,NG
94.      K=I+NG
95.      PRINT 3E,I,(NOD(I,J),J=1+4),K,(NOD(K,J),J=1+4)
96.      23  CONTINUE
97.      3F  FORMAT (E10.4X,5T6)
98.      IF (NG-NC1)59,197,75
99.      75  PRINT 7010,NG1,(NOD(ING1,J),J=1+4)
*****GENERATE NODAL COORDINATES*****
100.      197  READ 1002,(RADN(I),I=1+NC1)
101.      1002  FORMAT (5E6.3)
102.      TF (NE-NS1124+193+137
103.      137  IF (NS1193+124+194
104.      138  NE=NF-1
105.      139  DO 196 I=1,NTNN
106.      NODUM=1+(T-1)*NF+1
107.      RAD(I)=RADN(NODUM)
108.      196  CONTINUE
109.      NF=NF+1
110.      TF (NS1201+201+202
111.      201  READ 1002,(TTAN(I),I=1+NF)

```

113. DO TO 206  
 114. 207 NSENS  
 115. DUMETEN./HNS  
 116. DO 203 I=1,NE  
 117. AIET  
 118. 203 TTAN(I)=AI-1,I=0JM  
 119. 204 DO 195 I=1,NE  
 120. TT=T+NE+INC-1  
 121. TDOUMETTAN(I)=.0174533  
 122. DO 192 JET,LINE  
 123. 192 TTA(J)=ETDUM  
 124. 193 CONTINUE  
 125. PRINT 191E, (RADN(I),I=1,NC)  
 126. 191E FORMAT (13H,RADIAL COORDINATES/(GX,AF1I,2))  
 127. PRINT 191E, (TTAN(I),I=1,NE)  
 128. 191E FORMAT (33H,UTRCUMFERENTIAL COORDINATES (DEGREES)  
 129. 1/IEX,AF1I,2))  
 130. \*\*\*\*\*INPUT MAGNITUDES OF CONCENTRATED LOADS IF ANY\*\*\*\*\*  
 131. TF (NNCL12001,2001,2000  
 132. 2000 READ 1002,MR,NODC(MR), (LOAD(MR,I),I=1,2)  
 133. 1002 FORMAT (T2,T3,2F10.1)  
 134. TF (MR-NNCL12000,2001,2001  
 135. \*\*\*\*\*INPUT VALUES OF SPECIFIED DISPLACEMENTS\*\*\*\*\*  
 136. 2001 IX=0  
 137. 2004 IX=IX+1  
 138. READ 1004,NODP(TX),(DISP(IX+I),I=1,2),JJ,IT  
 139. 1004 FORMAT (T5,2F5.1,I3,I2)  
 140. NODM=IX  
 141. TF (TX-NNSD1253,2002,2002  
 142. 202 TF (JJ)2004,2004,2004  
 143. 204 DO 206 I=1,30  
 144. IRET=TI+NODP(NODM)  
 145. IX=IX+1  
 146. NODP(IX)=IR  
 147. DO 205 J=1,2  
 148. 205 DISP(IX,J)=DISP(IXUM,J)  
 149. 206 CONTINUE  
 150. IF (IX-NNSD12104,7002,2002  
 151. 2003 NLSE=100  
 152. \*\*\*\*\*INITIALIZE LOAD AND STIFFNESS MATRICES TO ZERO\*\*\*\*\*  
 153. KME=ANL0  
 154. DO 107 I=1,KM  
 155. F(I)=0.  
 156. DO 103 J=1,NE  
 157. 103 S(J)=0.  
 158. \*\*\*\*\*GENERATE GLOBAL MATRICES S(I,J) AND F(I) IN BLOCKS,  
 159. TREATING TWO BLOCKS EACH TIME \*\*\*\*\*  
 160. C SEARCH FOR ELEMENT THAT CONTRIBUTES TO UPPER BLOCK  
 161. C AND GENERATE ITS STIFFNESS AND LOAD MATRICES  
 162. READND 16  
 163. REWIND 12  
 164. LYNE=1  
 165. NENED  
 166. 104 NGEN=NRL+1  
 167. NLE=(NRM-1)\*LLE+1  
 168. NTEN=NRL+3  
 169. NTENT=LLE

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```

170.      DO 105 T=1,NELEM
171.      IF (NOD(T,1))105,105,105
172.      106 TN=T
173.      DO 107 J=1,4
174.      IF (NOD(T,J)*2-NT)108,108,107
175.      107 CONTINUE
176.      GO TO 105
177.      I1=NOD(I,1)
178.      I2=NOD(I,3)
179.      X1=.5*(RAD(I2)+RAD(I1))
180.      X2=.5*(RAD(I2)-RAD(I1))
181.      Y2=.5*(TTA(I2)-TTA(I1))
182.      TF (Y2)206,206,207
183.      206 Y2=.5*(S.28719-TTA(I1))
184.      207 Y1=TTA(I1)+Y2
185.      F=P0(I,1)
186.      PR=PD(I,2)
187.      CCR=PD(I,3)
188.      WRITE (12) IN,X1,X2,Y1,Y2,E,PR,CCR
189.      LYNC=(I-1)/NFR
190.      TF (LYNC-LYN)114,110,114
191.      114 LYN=LYNC
192.      CALL INTFG
193.      C      ADD ELEMENT STIFFNESS MATRIX INTO UPPER
194.      C      TRIANGULAR PORTION OF S(I,J)
195.      110 DO 219 K=1,4
196.      DO 219 J=1,2
197.      KN=(K-1)*2+J
198.      KM=(NOD(I,K1)-1)*2+J-(NFN-1)*NLB
199.      DO 213 L=1,4
200.      IF (NOD(I,L1)-NOD(I,K1))213,214,214
201.      214 DO 215 MX=1,2
202.      LN=(L-1)*2+MX
203.      LM=(NOD(I,L1)-NOD(I,K1))*2+MX-J+1
204.      IF (LM)215,215,212
205.      212 S(KM,LM)=S(KM,LM)+SL(KN,LN)
206.      215 CONTINUE
207.      217 CONTINUE
208.      219 CONTINUE
209.      NOD(T,1)=NOD(T,1)
210.      105 CONTINUE
211.      *****INTRODUCE CONCENTRATED LOADS TO F(I)*****
212.      IF (NNCL)10,10,9
213.      9 DO 4 I=1,NNCL
214.      IF (NODC(I)*2-NL)14,6,5
215.      5 IF (NODC(I)*2-NT)16,6,4
216.      6 DO 7 J=1,2
217.      IF (CLOAD(T,J))18,7,8
218.      8 KM=(NODC(I)-1)*2+J-(NFN-1)*NLB
219.      F(KM)=F(KM)+CLOAD(I,J)
220.      7 CONTINUE
221.      4 CONTINUE
222.      *****INTRODUCE BOUNDARY CONDITIONS TO S(I,J) AND F(I)*****
223.      10 DO 14 I=1,NNSD
224.      IF (NODB(I)*2-NL)14,16,15
225.      15 IF (NODB(I)*2-NT)16,16,17
226.      16 DO 15 J=1,2

```

227. TF (100.-ABS(DISP(I,J))18.18.25  
 228. 25 KM=(NODB(I)-1)\*2+J-(NBN-1)\*NLR  
 229. F(KM)=DISP(I,J)  
 230. S(KM+1)=1.  
 231. KK=KM  
 232. DO 19 K=2,NRW  
 233. KK=KK+1  
 234. F(KK)=F(KK)-S(KM,K)+F(KM)  
 235. 19 S(KM,K)=0.  
 236. KK=KM  
 237. DO 20 MX=2,NBW  
 238. KK=KK-1  
 239. TF (KK)21.21.22  
 240. 22 F(KK)=F(KK)-S(KK,MX)+DISP(I,J)  
 241. 20 S(KK,MX)=0.  
 242. 21 CONTINUE  
 243. 18 CONTINUE  
 244. GO TO 14  
 245. 17 KRK=NODB(I)\*2-NT-NBW  
 246. TF (KRK)23.23.14  
 247. 23 DO 24 J=1,2  
 248. TF (100.-ABS(DISP(I,J))24.24.26  
 249. 25 KM=(NODB(I)-1)\*2+J-(NBN-1)\*NLB  
 250. LRL=KM+1-NBW-NLB  
 251. TF (LRL)28.28.24  
 252. 28 KK=NLB+1  
 253. LIM1=KM-NLB+1  
 254. DO 27 ML=LIM1,NRW  
 255. KK=KK-1  
 256. F(KK)=F(KK)-S(KK,ML)+DISP(I,J)  
 257. S(KK,ML)=0.  
 258. 27 CONTINUE  
 259. 24 CONTINUE  
 260. 14 CONTINUE  
 261. C\*\*\*\*\*WRITE UPPER BLOCK ON TAPE AND SHIFT UP LOWER BLOCK\*\*\*\*\*  
 262. WRITE (101) (FIN), (S(N,M), M=1,NBW), N=1,NLB  
 263. IF (NTNN\*2-NLB)EQ.0,31  
 264. 31 IF (NTNN\*2-NT)EQ.50,50,32  
 265. 32 DO 33 L=1,NLB  
 266. LL=L+NLB  
 267. F(LL)=F(LL)  
 268. FILLI=0.  
 269. DO 33 ML=1,NBW  
 270. S(LL,ML)=S(LL,ML)  
 271. 33 S(LL,ML)=0.  
 272. GO TO 104  
 273. C\*\*\*\*\*COMPUTE AND PRINT NODAL DISPLACEMENTS\*\*\*\*\*  
 274. 50 CALL DISPL  
 275. IF (NLRI)98.998,138  
 276. 138 DO 480 MJ=1,NCOU  
 277. PRINT 2999,MJ  
 278. 2999 FORMAT (10H0CYCLF NO.,T3)  
 279. CRLF=10000000.  
 280. IF (NCHO)96.95.96  
 281. 95 PRINT 3000  
 282. 3000 FORMAT (38HU STRESSES AT ELEMENT CENTRAL POINT)  
 283. PRINT 3002

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284.    *H112 FORMAT (8H ELEMENT,7H RADIAL,6X,EHCTRUM,6X,EH SHEAR,
285.    1GX,EHPRNCP1,6X,6HFRNCP2,6X,GHMAXSHR,6X,EHLD FAC)
286.    OR REWIND 12
287.    DO 465 MI=1,NFLEM
288.    READ (12) IN,X1,X2,Y1,Y2,E,PR,CCR
289.    IF (INOD(IN,4) NE 52,52
290.    S2 NOD(IN,1)=APS(NGD(IN,1))
291.    TCR=-CCR*CTRAT
292.    SCR=-CCR
293.    DO 466 J=1,4
294.    KP=1+(NOD(IN,J)-2-1)/NLB
295.    L=(NOD(IN,J)-1-2-NLB)*(KP-2)
296.    LTM2=2+J
297.    LTM1=LTM2-1
298.    DO 468 K=LTM1,LTM2
299.    L=L+1
300.    NUE F(K)=S(L,KP)
301.    468 CONTINUE
302.    DO 81 I=1,4
303.    DL(I)=0.
304.    DL(I+4)=0.
305.    DO 80 K=1,7,2
306.    KK=K+1
307.    K1=KK/2
308.    DL(T)=DL(T)+C(I,K1)*F(K)
309.    80 DL(T+4)=DL(T+4)+C(I,K1)*F(KK)
310.    81 CONTINUE
311.    X(1)=GA(E)
312.    Y(1)=GB(E)
313.    M=1
314.    N=1
315.    CALL MATR
316.    DO F1 I=1,3
317.    Z(I)=U.
318.    DO FU J=1,8
319.    Z(I)=Z(I)+BB(I,J)*DL(J)
320.    E1 CONTINUE
321.    DUM1=SOR(T,25*(Z(1)-Z(2))+2,Z(3)+Z(1))
322.    DUM2=.5*(Z(1)+Z(2))
323.    TS=DUM2+DUM1
324.    CS=DUM2-DUM1
325.    SS=DUM1
326.    Z2(IN,1)=Z(1)
327.    Z2(IN,2)=Z(2)
328.    Z2(IN,3)=Z(3)
329.    Z2(IN,4)=TS
330.    Z2(IN,5)=CS
331.    Z2(IN,6)=SS
332.    TR=1.0E-5
333.    IF (TS-TR162.02.E3
334.    E3 IF (CS-ZR165.E7.E7
335.    E5 IF (CCR/CS-TCR/TS1E2.67.67
336.    E7 HLF=TCR/TS
337.    KEF(IN)=I
338.    CO TO E4
339.    E2 HLF=CCR/CS
340.    KEF(IN)=0

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```

341. 64 TF (ABS(SSI)-ZRIEP,F9.8F
342. 8E TF (ABS(SCR/SSI)-HLF)E8.E9.E3
343. F8 HLF=ABS(SCR/SSI)
344. KFF( TN)=1
345. 69 FCR=1000.0HLF
346. FLOF(INI)=HLF
347. TF (NCH01) 97.98.97
348. 96 PRINT 4002,TN,Z(1),Z(2),Z(3),TS,CS,SS,HLF
349. 4002 FORMAT (IE,BF12.5)
350. 37 TF (HLF-CRLF) 6E+465.465
351. FE CRLF=HLF
352. FCR=CRC
353. ND=TN
354. 4EE CONTINUE
355. TF (NCH01)E+122+125
356. 125 PRINT 12E,FCRG
357. 126 FORMAT (17HCRITICAL LOAD = ,F12.5)
358. 122 NBK=NLCL-1+2/NL8+1
359. L8K=NLCL+2-1-(NBK-1)*NL8
360. MBK=NLB+LBK
361. DSF=FCRG/1000.+S(MBK,NBK)
362. PRINT 400E,DSF
363. 4005 FORMAT (72H)VERT. DTSPL. AT POINT OF LOAD =,F12.5)
364. NBK1=(NST1-1)+2/NL8+1
365. NBK2=(NST2-1)+2/NL8+1
366. LBK1=NST1+2-1-(NEK1-1)*NL8
367. LBK2=NST2+2-1-(NPK2-1)*NL8
368. MBK1=NLB+LBK1
369. MBK2=NLB+LBK2
370. STR=(S(MBK2,NBK2)-S(MBK1,NBK1))/IRAD(NST2)
371. 1-RAD(NST1))+FCRG/1000.
372. PRINT 41B2,STR
373. 413E FORMAT (20H)STRATN AT CENTER = ,F12.5)
374. JEN=0
375. DU 467 MK=1.NELFM
376. TF (NODIMK,4114E7,4E7,7E
377. 7E TRUN-(FLDFIND1)-FLDF(MK1)/FLDF(ND1)
378. IF (ARS(STRUN1-TRN)77,77,467
379. 77 JFN=JFN+1
380. NFL(JEN)=MK
381. DO 468 MC=1.IRN
382. TF (MK-NTK(MC))4EB+7B+468
383. 78 NAME=MK
384. GO TO 907
385. 468 CONTINUE
386. 487 CONTINUE
387. TF (NCH01)94,221,04
388. 220 PRINT 4001
389. 4001 FORMAT (75H)THE FOLLOWING ELEMENTS HAVE FAILFD
390. CO TO 132
391. 94 PRINT 3003
392. 3003 FORMAT (34H) STRESSES AT FAILFD ELEMENTS)
393. PRINT 3004
394. 3004 FORMAT (18H ELEMENT,7H RADIAL,EX,EH CIRCUM,EX,EH SHAR,
395. 10X,EH PRNCP1,6X,EH PRNCP2,6X,EH MAXSHR,6X,12H FAILURE MODE)
396. 132 DO 469 ND=1,JFN
397. ND=NFL(MD)

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```

398.    TF (NCHO)127,124,123
400.    123 IF (KEE(ND)1292+28E+284
401.    284 PRINT 287,ND,122(ND,I1,I=1,6)
401.    287 FURMATT(IF,EF12.5,5X,7HTENSTONI)
401.    GO TO 93
402.    286 PRINT 288,ND,127(ND,I1,I=1,6)
403.    288 FORMAT(IF,EF12.5,5X,11HCOMPRESSSION)
404.    GO TO 92
405.    282 PRINT 289,ND,127(ND,I1,I=1,6)
406.    289 FORMAT(IF,EF12.5,5X,5HSHEAR)
407.    GO TO 93
408.    120 IF (KEF(ND)192+28+84
409.    94 PRINT 87,ND
410.    87 FORMAT (EX,T2,13H - IN TENSIONI
411.    GO TO 93
412.    88 PRINT 88,ND
413.    89 FORMAT (FX,T3,17H - IN COMPRESSIONI
414.    GO TO 93
415.    92 PRINT 89,ND
416.    93 FORMAT (EX,T2,11H - IN SHEAR)
417.    93 NNR=1
418.    TF (NTNN+2-NLR149+49+42
419.    42 LYNN=NTNN
420.    DO 411 I=1+4
421.    IF (INOD(ND,I)-LYNN)141+4D+46
422.    41 LYNN=NOD(ND,I)
423.    40 CONTINUE
424.    NNS=FLYNN-11+3/NL8+1
425.    IF (INNE-NSN)144+49+44
426.    44 MILA=NBN-NNP
427.    DO 47 I=1,MILA
428.    BACKSPACE 11
429.    47 CONTINUE
430.    N2=NLR+2
431.    N1=NLB+1
432.    READ (101) (FIN),IS(N,M),M=1,NBU1+N=1+N2
433.    BACKSPACE 11
434.    49 BACKSPACE 11
435.    READ (101) (FIN),IS(N,M),M=1,NBU1+N=1,NLB1
436.    T1=NOD(ND+1)
437.    T2=NOD(ND+2)
438.    X1=E+F*(RAD(T2)+RAD)(T1))
439.    Y2=E+F*(RAD(T2)-RAD(T1))
440.    Y2=F*(TTA(T2)-TTA(T1))
441.    TF (Y21294+294+295
442.    294 Y2=E+F*(E+2B717-TTA(T1))
443.    295 Y1=TTA(T1)+Y2
444.    F=PD(ND+1)
445.    PR=PD(ND+2)
446.    CALL INTFG
447.    DO 48 I=1,4
448.    DO 47 J=1,NNSD
449.    TF (NOD(ND,T1)-NOD(J,147+46+47
450.    46 DO 47 K=1,2
451.    TF (DISP(J,K)-100,170+45+45
452.    70 KP=(T-1)+2+K
453.    DU 71 L=1,B
454.

```

SL(KP,L)=0.  
SL(L,KP)=0.  
71 CONTINUE  
45 CONTINUE  
47 CONTINUE  
48 CONTINUE  
DO 79 K=1,4  
DO 79 J=1,2  
KNE=(K-1)\*2+J  
KME=(NOD(ND,K)-1)\*2+J-(NNB-1)\*NL6  
DO 79 L=1,4  
TF (NOD(ND,L)-NOD(ND,K1))=173 OR 74  
79 DO 79 MX=1,2  
LN=(L-1)\*2+MX  
LM=(NOD(ND,L)-NOD(ND,K1))\*2+MX-J+1  
IF (LM)75,7F,72  
72 S(KP,LM)=S(KM,LM)+.9\*SL(KN,LM)  
75 CONTINUE  
73 CONTINUE  
79 CONTINUE  
BACKSPACE 11  
WRTTF (10) (F(N),IS(N,M),M=1,NBN),N=1,NLB)  
IF (NNB-NBN)82,B7,S2  
82 WRTTF (10) (F(N),IS(N,M),M=1,NBN),N=N1,N2)  
BACKSPACE 11  
DO 175 KE=1,MILA  
READ (10) (F(N),IS(N,M),M=1,NBN),N=1,NLB)  
176 CONTINUE  
52 NOD(ND,4)= -NOD(ND,4)  
469 CONTINUE  
CALL DISPL  
IF (NLB)998,992,430  
430 CONTINUE  
GO TO 998  
957 PRINT 25E,NAMCHA  
29C FORMAT (1BH FLMFNT,TU,  
14H WHICH IS IN VICINITY OF LOAD HAS FAILED/  
11H RUN TS TERMINATED)  
958 STOP  
FNO

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END OF COMPILETIME: NO DIAGNOSTICS.

1. SUBROUTINE DISPL  
 COMMON/COLOR/NROW,MM,NUMBLK,NFG,B12UB,A12UB,I001  
 2. 7R=1,UE-E  
 3. NN=NFG  
 4. NF=1  
 5. NH=NN+NN  
 6. MT=NFQ  
 7. ML=NN+1  
 8. NU=1  
 9. IF (NUMPLK-1)SI+1EE+91  
 10. 11 REWIND 11  
 11. REWIND 10  
 12. NSE=0  
 13. GO TO 160  
 14. C REDUCE EQUATIONS BY BLOCKS  
 15. C 1. SHIFT BLOCK OF EQUATIONS  
 16. 160 NB=NR+1  
 17. DO 125 N=1,NN  
 18. NM=NN+NN  
 19. B(N)=B(NM)  
 20. B(NM)=0.  
 21. DO 125 M=1,MM  
 22. A(N,M)=A(NM,M)  
 23. 125 A(NM,M)=0.  
 24. C 2. READ NEXT BLOCK INTO CORE  
 25. TF (NUMBLK-NB)1EU+1EE+1FU  
 26. 150 READ (1H) (B(N), (A(N,M), M=1,MM), N=ML,NH)  
 27. IF (NB)1E4,100,154  
 28. C 3. REDUCE BLOCK OF EQUATIONS  
 29. 155 MTENROW+2-(NUMPLK-1)\*NFG  
 30. NF=0  
 31. 154 DO 380 N=1,MT  
 32. C CHECK FOR VERY SMALL OR ZERO ELEMENT ON DIAGONAL  
 33. CH=0.  
 34. DO 165 M=1,MM  
 35. IF (CH-ABS(A(N,M)))194,165,166  
 36. 166 CHEART(A(N,M))  
 37. 165 CONTINUE  
 38. XXX=ABS(A(N,1))/CH  
 39. IF (XXX-ZR)1E-5E+259  
 40. C IF ZERO OR SMALL ON DIAGONAL TERMINATE RUN  
 41. 155 PRINT 230,XXX,N,NR  
 42. 230II FORMAT (79H,SMALL DIAGONAL - A(N,1)/MAX A(N,M) IS,  
 43. 1F14.0,12H TN EQUATION, I4,SH OF BLOCK, T4)  
 44. NF=0  
 45. GO TO 50  
 46. C NORMAL ELIMINATION PROCEDURE  
 47. 200 B(N)=B(N)/A(N,1)  
 48. DO 375 L=2,MM  
 49. IF (ABS(A(N,L))-ZR)375,375,376  
 50. 376 C=A(N,L)/A(N,1)  
 51. L=N+L-1  
 52. J=0  
 53. DO 370 K=L,MM  
 54. J=J+1

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56. 350 A(I,J)=A(I,J)-C\*A(N,K)  
 57. B(I)=B(I)-A(N,L)\*B(N)  
 58. A(N,L)=C  
 59. 375 CONTINUE  
 60. 760 CONTINUE  
 61. C WRITE BLOCK OF EQUATIONS ON TAPE 11  
 62. IF (NUMBLK-NB) 84, 410, 54  
 63. 94 WRITE (11)(B(N),A(N,M),M=1,MM),N=1,NN  
 64. GO TO 100  
 65. C BACK-SUBSTITUTION  
 66. 410 DO 450 M=1,MT  
 67. N=M+1-M  
 68. L=I  
 69. 87 DO 425 K=L,MM  
 70. L=N+K-1  
 71. 425 B(N)=B(N)-A(N,K)\*B(L)  
 72. 450 CONTINUE  
 73. M=N+1  
 74. DO 460 N=1,MT  
 75. NM=N+NN  
 76. A(NM,NP)=B(N)  
 77. IF (NB-NUMBLK) 710, 460, 450  
 78. 710 A(NM,M)=B(NM)  
 79. 460 B(NM)=B(N)  
 80. NB=NB-1  
 81. IF (NB) 500, 500, 71  
 82. 71 BACKSPACE 11  
 83. MT=NN  
 84. READ (11)(B(N),A(N,M),M=1,MM),N=1,NN  
 85. BACKSPACE 11  
 86. GO TO 400  
 87. END K=1  
 88. NM=NEG-1  
 89. DO 600 I=1,NROW  
 90. NM=NM+3  
 91. NMF=NMT+1  
 92. IF (NMF-2.\*NFH1E02+EU2+EU1  
 93. 601 KEK+1  
 94. NMT=NEG+1  
 95. NMF=NM+1  
 96. END CONTINUE  
 97. END CONTINUE  
 98. 80 RETURN  
 99. END  
 100. \*\*\*\*=  
 101. \*\*\*\*=

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END OF COMPILEATION: NO PTAGNOSTICS.

```

1.      SUBROUTINE MATE
2.      COMMON/JACKP/C(4,4),R(7,8),FL(8),SL(8,8),RR(3,8)
3.      COMMON/BGFAP/PR,E,T,TCR,CCR
4.      COMMON/KRING/X1,X2,Y1,Y2,X(3),Y(3),Z(3),M,N
5.      RR=EX1*X2*X(M)
6.      DUM=F*T/(1.-PR*PR)
7.      XX=EX(M)
8.      YY=Y(N)
9.      R(1,2)=1./X2
10.     R(1,4)=YY/X2
11.     R(2,1)=1./RR
12.     R(2,2)=XX/RR
13.     R(2,3)=YY/RR
14.     R(2,4)=XX+YY/RR
15.     R(2,7)=1./RR/Y2
16.     R(2,8)=XX/RR/Y2
17.     R(3,2)=R(2,7)
18.     R(3,4)=R(2,9)
19.     R(3,5)=-R(2,1)
20.     R(3,6)=1./X2-XX/RR
21.     R(3,7)=-R(2,2)
22.     R(3,8)=YY+R(3,6)
23.     DO 25 T=1,8
24.     RR(1,I)=DUM*(R(1,I)+PR*R(2,I))
25.     RR(2,I)=DUM*(R(2,I)+PR*R(1,I))
26.     RR(3,I)=DUM*.5*(1.-PR)*R(3,I)
27.   25 CONTINUE
28.   RETURN
29. END

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END OF COMPILEATION: NO DIAGNOSTICS.

```

1.      SUBROUTINE TNTFG
2.      COMMON/JACKF/C(4,4),R(7,8),FL(81,SL(8,8),BR(3,8)
3.      COMMON/KRINF/X1,X2,Y1,Y2,X(7),Y(3),Z(3),M,N
4.      DIMENSION H(3),AA(3),DL(8),D(3,8),DX(3,8)
5.      DATA (H(I,I),I=1,7)/ .55555555, .98888888, .55555555/
6.      DATA (AA(I,I),I=1,3)/ - .775, 0, .775 /
7.      DO 3 T=1,3
8.      DL(T)=0.
9.      FL(T)=0.
10.     DO 4 K=1,3
11.     DX(T,K)=0.
12.     DO 5 J=T,8
13.     D(I,J)=0.
14.     SL(J,I)=0.
15.     3 CONTINUE
16.     DO 31 L=1,3
17.     M=L
18.     X(M)=AA(M)
19.     RR=X1+X2*X(M)
20.     DO 30 J=1,3
21.     N=J
22.     Y(N)=AA(N)
23.     CALL MATP
24.     DUM1=H(L)*H(J)+RR*X2*Y2
25.     DO 20 NROW=1,8
26.     DO 20 NCOL=NROW,8
27.     DUM2=0.
28.     DO 100 KK=1,3
29.     DUM2=DUM2+B(KK,NROW)*BR(KK,NCOL)
30.     D(NROW,NCOL)=D(NROW,NCOL)+DUM2*DUM1
31.     20 CONTINUE
32.     31 CONTINUE
33.     DO 6 I=1,7
34.     J=I+1
35.     DO 8 K=J,8
36.     D(K,T)=D(I,K)
37.     I1=-1
38.     DO 29 I=1,8
39.     T1=I1+1
40.     T2=(T-1)/2+1
41.     DO 24 J=1,3
42.     DO 23 K=1,4
43.     K1=K+I1*4
44.     23 DX(I,J)=DX(T,J)+(C(K,T2)*D(K1,J))
45.     24 CONTINUE
46.     TF (T1-1)*2+21,21
47.     21 T1=-1
48.     26 CONTINUE
49.     T1=-1
50.     DO 28 I=1,8
51.     I1=I1+1
52.     T2=(T-1)/2+1
53.     DO 27 J=T,8
54.     DO 26 K=1,4
55.     K1=K+I1*4

```

```
55.    2E SL(J,I)=SL(J,T)+DX(J,K1)*CX(K,T2)
56.    27 CONTINUE
57.    IF (T1-I)GE22+72
58.    22 T1=-1
59.    28 CONTINUE
60.    DO 7 I=1,7
61.    J=T+1
62.    DO 7 K=J,9
63.    7 SL(I,K)=SL(K,I)
64.    RETURN
65.    END
```

END OF COMPIILATION: NO DIAGNOSTICS.

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```
1.      SUBROUTINE UNIFORM(A,B,C)
2.      D=RNUN(R)
3.      C=A+(B-A)*D
4.      RETURN
5.      END
```

END OF COMPILEATION: NO DIAGNOSTICS.

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